

# IDEALISTIC EXPONENTS AND THEIR CHARACTERISTIC POLYHEDRA

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**ABSTRACT.** In this paper we study Hironaka’s idealistic exponents in the situation over  $\text{Spec}(\mathbb{Z})$ . In particular we give an idealistic interpretation of the tangent cone, the directrix, and the ridge. The main purpose is to introduce the notion of characteristic polyhedra of idealistic exponents and deduce from them intrinsic data on the idealistic exponent.

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## INTRODUCTION

Polyhedra are important tools to study the nature of singularities. For example, in [H3] Hironaka used characteristic polyhedra of singularities to prove resolution of excellent hypersurface singularities in dimension two. Further he introduced the notion of characteristic polyhedra of an ideal in [H1], which in [CJS] has been used to extend [H3] to the case of excellent schemes of dimension at most two. Moreover, the characteristic polyhedron of an ideal plays an essential role in the work of Cossart and the author [CSc2], where a strictly decreasing invariant for the strategy of [CJS] is constructed.

Based on Hironaka’s work we develop in this paper the notion of characteristic polyhedra of idealistic exponents. The starting point for this study was the task to show that the invariant introduced by Bierstone and Milman [BM1] in order to prove constructive resolution of singularities in characteristic zero can be purely determined by considering certain polyhedra and their projections. This result and thus a first application of characteristic polyhedra of idealistic exponents is discussed in [Sc2].

Nevertheless the theory of these polyhedra goes beyond the situation over fields of characteristic zero. Another interesting application would be the reinterpretation of the strategy of [CJS] in the language of idealistic exponents. Since Hironaka’s characteristic polyhedron is intensively used in [CJS] and [CSc2] there is the need to introduce appropriate polyhedra in the setting of idealistic exponents.

Another direction to go on would be to interpret the characteristic polyhedra of idealistic exponents in the language of Rees algebras which are used by Villamayor and his students to study singularities over perfect fields ([BrV], [BeV], [BGV]), or in the language of idealistic filtrations by Kawanoue and Matsuki [K]. The investigations on the behavior of the polyhedra in this theories might give new insight in their approaches.

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Given a regular scheme  $Z$  of finite type over  $\text{Spec}(\mathbb{Z})$  we recall the notion of pairs  $\mathbb{E} = (J, b)$ , where  $J \subset \mathcal{O}_Z$  is a quasi-coherent ideal sheaf and  $b \in \mathbb{Z}_+$  an integer. (Later this will be extended to  $b \in \mathbb{Q}_+$ ). Roughly speaking, two pairs are equivalent, denoted by  $\sim$ , if they under go the same resolution process. Additionally, there is a technical part in the definition of  $\sim$  that the first condition is even true after adding some new variables. The latter is a crucial property for proofs and is sometimes called Hironaka's trick. An idealistic exponent  $\mathbb{E}_\sim$  denotes then the equivalence class of a pair  $\mathbb{E}$  with respect to  $\sim$ .

To a pair  $\mathbb{E}$  we can associate the tangent cone, the directrix and the ridge. The latter two are closely related: for example if the base field is perfect, then the reduced ideal of the ridge and the ideal of the directrix coincide. A detailed discussion of this is given in Remark 2.6.

It was already shown in [H2] that the directrices of equivalent pairs coincide (perfect base field!). In order to reveal a similar relation for the tangent cone resp. the ridge we introduce the concepts of *idealistic tangent cones*  $\mathbb{T}_x(\mathbb{E})$ , *idealistic directrices*  $\text{Dir}_x(\mathbb{E})$ , and *idealistic ridges*  $\text{Rid}_x(\mathbb{E})$  of a pair  $\mathbb{E}$  at a singular point  $x$ . Whereas the first concept (of an idealistic variant of the tangent cone) already appears in the work of Benito and Villamayor (see section 1.2 in [BeV]) or Kawanoue and Matsuki (see Definition 1.1.2 in [K]), the latter two are completely new. We have

**Proposition A.** *Let  $\mathbb{E}_1 \sim \mathbb{E}_2$  be two equivalent pairs and  $x \in \text{Sing}(\mathbb{E}_1) = \text{Sing}(\mathbb{E}_2)$ . Then*

$$\mathbb{T}_x(\mathbb{E}_1) \sim \mathbb{T}_x(\mathbb{E}_2), \quad \text{Dir}_x(\mathbb{E}_1) \sim \text{Dir}_x(\mathbb{E}_2), \quad \text{and} \quad \text{Rid}_x(\mathbb{E}_1) \sim \text{Rid}_x(\mathbb{E}_2).$$

This means the idealistic tangent cone, the idealistic directrix and the idealistic ridge are actually invariants of the idealistic exponent  $\mathbb{E}_\sim$ .

Another important tool for the study of singularities is the coefficient ideal with respect to  $V(y)$ , where  $(y) = (y_1, \dots, y_r)$  is part of a regular system of parameters (short r.s.p.)  $(u, y) = (u_1, \dots, u_e, y)$  of the local ring  $R = \mathcal{O}_{Z,x}$  at a singular point  $x$ . We define its counterpart in the language of idealistic exponent, called *coefficient pairs*  $\mathbb{D}_x(\mathbb{E}, u, y)$ , where  $x \in \text{Sing}(\mathbb{E})$ . So far no special assumptions on the system  $(y)$  are made except for being part of a r.s.p. We then can show

**Proposition B.** *Let  $\mathbb{E}_1 \sim \mathbb{E}_2$  be two equivalent pairs on a regular local Noetherian ring  $R$  (e.g.  $R = \mathcal{O}_{Z,x}$ ) and let  $(u, y)$  be a r.s.p. for  $R$ .*

- (1) *Then the coefficient pairs with respect to the same  $V(y)$  are again equivalent,*

$$\mathbb{D}_x(\mathbb{E}_1, u, y) \sim \mathbb{D}_x(\mathbb{E}_2, u, y)$$

- (2) *If we have two choices  $V(y)$  and  $V(z)$  for a fixed system  $(u)$  and a fixed pair  $\mathbb{E}$  such that  $\mathbb{E} \cap (y, 1) \sim \mathbb{E} \cap (z, 1)$ , then the coefficient pairs are also equivalent,*

$$\mathbb{D}_x(\mathbb{E}, u, y) \sim \mathbb{D}_x(\mathbb{E}, u, z)$$

In particular,  $\mathbb{D}_x(\mathbb{E}, u, y)_\sim$  is an invariant of the idealistic exponent  $\mathbb{E}_\sim$ .

For fixed data  $(\mathbb{E}, u, y)$  we define its associated polyhedra  $\Delta(\mathbb{E}; u; y)$  and investigate their first properties. Unfortunately, they behave badly under the equivalence relation  $\sim$ ; in Example 4.9 we show that there exist equivalent pairs whose associated polyhedra are not equal. Thus the polyhedron is not an invariant of the idealistic exponent  $\mathbb{E}_\sim$ .

Nevertheless there is some hope: We can recover the order of the coefficient idealistic exponent from the associated polyhedra which is an invariant of  $\mathbb{E}_\sim$  (Proposition 4.7).

Imitating the construction of Hironaka's characteristic polyhedron for a singularity we define the characteristic polyhedron  $\Delta(\mathbb{E}; u)$  of a pair  $\mathbb{E}$  with respect to a certain system of regular elements  $(u) = (u_1, \dots, u_e)$ . More precisely,  $\Delta(\mathbb{E}; u)$  is the intersection over all possible choices for  $(y)$ ,

$$\Delta(\mathbb{E}; u) = \bigcap_{(y)} \Delta(\mathbb{E}; u; y).$$

An interesting question is then if there is a good choice for  $(y)$  such that  $\Delta(\mathbb{E}; u; y) = \Delta(\mathbb{E}; u)$ . We give an affirmative answer in

**Theorem C.** Let  $\mathbb{E} = (J, b)$  be a pair on a regular local Noetherian ring  $R$  and denote by  $(u, y) = (u_1, \dots, u_e; y_1, \dots, y_r)$  a r.s.p. for  $R$  such that the initial forms of  $(y)$  yield the whole directrix  $\text{Dir}_x(\mathbb{E})$ .

Then there exist elements  $(y^*) = (y_1^*, \dots, y_r^*)$  in  $\widehat{R}$  such that  $(u, y^*)$  is a r.s.p. for  $\widehat{R}$ ,  $(y^*)$  yields  $\text{Dir}_x(\mathbb{E})$ , and

$$\Delta(\mathbb{E}; u; y^*) = \Delta(\mathbb{E}; u)$$

Moreover, later we can give a simple proof that if  $V(y)$  has maximal contact then the polyhedron  $\Delta(\mathbb{E}; u; y)$  associated to a pair  $\mathbb{E}$  is independent of the choice of the maximal contact variables  $(y)$  (Proposition 6.1).

But still the characteristic polyhedra of pairs do not behave well under  $\sim$ . Thus we discuss in Remark 5.8 the concept of a unique characteristic polyhedron of an idealistic exponent  $\mathbb{E}_\sim$ . Further we sketch in Remark 5.9 how the notion of characteristic polyhedra can be extended to the quasi-homogeneous situation, i.e. where we put certain weights on each element of the r.s.p. of  $R$ .

For our purposes it is not crucial that we obtain a unique characteristic polyhedron for an idealistic exponent. The important issue is that the information which we obtain from the polyhedra are invariants of the idealistic exponent  $\mathbb{E}_\sim$ . In this context we prove for

$$\delta_x(\mathbb{E}, u) := \delta_x(\Delta(\mathbb{E}; u)) = \min\{|v| = v_1 + \dots + v_e \mid v \in \Delta(\mathbb{E}; u)\} \in \frac{1}{b!} \mathbb{Z}_+^e$$

**Theorem D.** The rational number  $\delta_x(\mathbb{E}, u)$  does not depend on  $(y)$  and is invariant under the equivalence relation  $\sim$ . Therefore  $\delta_x(\mathbb{E}, u)$  is an invariant of the idealistic exponent  $\mathbb{E}_\sim$  and  $(u)$ .

In the situation over fields of characteristic zero this will be the essential ingredient to deduce the connection between the invariant of Bierstone and Milman and the characteristic polyhedra of idealistic exponents in [Sc2].

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## 1. PAIRS AND IDEALISTIC EXPONENTS

Originally Hironaka introduced idealistic exponents on regular schemes which are of finite type over a perfect field, see [H2] and [H4]. Later he extended this notion to idealistic exponents on regular Noetherian schemes which are not necessarily of finite type over a base, see [H5]. In [Sc1] the author recalled the definitions and worked out the proofs of the first properties for the case over arbitrary fields in detail. In this article we focus on idealistic exponents on a regular Noetherian schemes of finite type over  $\mathbb{Z}$  which is sufficient for our purposes. (We follow the usual convention and write *over*  $\mathbb{Z}$  and not *over*  $\text{Spec}(\mathbb{Z})$ ).

Let  $Z$  be a regular irreducible scheme of finite type over  $\mathbb{Z}$ . Note that by the Hilbert basis theorem  $Z$  is Noetherian.

**Definition 1.1.** A pair  $\mathbb{E} = (J, b)$  on  $Z$  is a pair consisting of a quasi-coherent ideal sheaf  $J \subset \mathcal{O}_Z$  and a positive integer  $b \in \mathbb{Z}_+$ .

We define its order at a point  $x \in Z$  (not necessarily closed) as

$$\text{ord}_x(\mathbb{E}) = \begin{cases} \frac{\text{ord}_x(J)}{b} & , \text{ if } \text{ord}_x(J) \geq b \text{ and} \\ 0 & , \text{ else,} \end{cases}$$

where  $\text{ord}_x(J) = \sup\{d \in \mathbb{Z}_{\geq 0} \cup \{\infty\} \mid J_x \subseteq M_x^d\}$  (and  $M_x$  denotes the maximal ideal in the local ring at  $x$ ). Further we define the singular locus (or support) of  $\mathbb{E}$  as

$$\text{Sing}(\mathbb{E}) = \{x \in Z \mid \text{ord}_x(J) \geq b\}.$$

We denote the closed subscheme corresponding to  $J$  by  $X \subseteq Z$ .

If  $Z = \text{Spec}(R)$  is affine, then we also say  $\mathbb{E}$  is a pair on  $R$ .

**Definition 1.2.** Let  $\mathbb{E} = (J, b)$  be a pair on  $Z$ . A blow-up  $\pi : Z' \rightarrow Z$  with center  $D$  is called *permissible* for  $\mathbb{E}$ , if  $D$  is regular and  $D \subseteq \text{Sing}(\mathbb{E})$ . The transform of  $\mathbb{E}$  is then given by  $\mathbb{E}' = (J', b)$ , where  $J'$  is defined via  $J\mathcal{O}_{Z'} = J'H^b$ , where  $H$  denotes the ideal sheaf of the exceptional divisor.

In other literature exist different notions of permissible centers, see for example [CJS]. There these are regular subschemes  $D \subset X$  such that additionally  $X$  is normally flat along  $D$  ([CJS], Definition 2.1).

**Definition 1.3.** We define a *local sequence of regular blow-ups (short LSB)* over  $Z$  as a sequence of the form

$$(1.1) \quad \begin{array}{ccccccc} Z = Z_0 \supset U_0 & \xleftarrow{\pi_1} & Z_1 \supset U_1 & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_{l-1}} & Z_{l-1} \supset U_{l-1} \xleftarrow{\pi_l} Z_l \\ & \cup & \cup & & & & \cup \\ & D_0 & D_1 & \dots & & & D_{l-1} \end{array}$$

where  $l \in \mathbb{Z}_+ \cup \{\infty\}$ , each  $U_i \subset Z_i$  is an open subscheme,  $D_i \subset U_i$  is a regular closed subscheme and  $\pi_{i+1} : Z_{i+1} \rightarrow U_i$  denotes the blow-up with center  $D_i$ ,  $0 \leq i \leq l-1$ .

**Remark 1.4.** Let  $\mathbb{E} = (J, b)$  be a pair on  $Z$  and consider a LSB as in (1.1). In Definition 1.2 we have introduced when a blow-up is permissible for  $\mathbb{E}$  and further we have defined the transform of  $\mathbb{E}$  under such a blow-up. Denote by  $\mathbb{E}_i$  the transform of  $\mathbb{E}_0 := \mathbb{E}$  in  $Z_i$  for  $0 \leq i \leq l-1$ . Then we say that the LSB (1.1) is permissible for  $\mathbb{E}$  if each blow-up  $\pi_{i+1}$  is permissible for  $\mathbb{E}_i$  for  $0 \leq i \leq l-1$ .

Let  $(t) = (t_1, \dots, t_a)$  be a finite system of indeterminates. Then we use the notation

$$Z[t] := Z \times_{\mathbb{Z}} \mathbb{A}_{\mathbb{Z}}^a = Z \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathbb{Z}[t]).$$

We consider the pair  $\mathbb{E}[t] = (J[t], b)$ , where  $J[t] = J\mathcal{O}_{Z[t]}$  (with respect to the canonical projection).

**Definition 1.5.** Let  $\mathbb{E}_1 = (J_1, b_1)$  and  $\mathbb{E}_2 = (J_2, b_2)$  be two pairs on  $Z$ . Then we define

$$\mathbb{E}_1 \subset \mathbb{E}_2$$

if the following condition holds:

$$(1.2) \quad \begin{array}{l} \text{Let } (t) = (t_1, \dots, t_a) \text{ be an arbitrary finite system of indeterminates and let } \mathbb{E}_i[t] = \\ (J_i[t], b_i), i \in \{1, 2\}. \text{ If any LSB over } Z[t] \text{ is permissible for } \mathbb{E}_1[t], \text{ then it is also} \\ \text{permissible for } \mathbb{E}_2[t]. \end{array}$$

Further we say  $\mathbb{E}_1$  and  $\mathbb{E}_2$  are *equivalent*,

$$\mathbb{E}_1 \sim \mathbb{E}_2,$$

if both  $\mathbb{E}_1 \subset \mathbb{E}_2$  and  $\mathbb{E}_1 \supset \mathbb{E}_2$ . By  $\mathbb{E}_1 \cap \mathbb{E}_2 \sim \mathbb{E}_3$  we mean that a LSB over  $Z[t]$  is permissible for  $\mathbb{E}_3[t]$  if and only if it is permissible for  $\mathbb{E}_1[t]$  and  $\mathbb{E}_2[t]$ .

An idealistic exponent  $\mathbb{E}_{\sim}$  is the equivalence class of a pair  $\mathbb{E}$ .

In other literature pairs are sometimes also called idealistic exponents (e.g. [H5]). In order to avoid confusion when coming to results and the dependence on the choice of a representant of the equivalence class, we use the original terminology [H2] of pairs and idealistic exponents.

By definition we have for  $x \in \text{Sing}(\mathbb{E}_1 \cap \mathbb{E}_2)$ :  $\text{Sing}(\mathbb{E}_1 \cap \mathbb{E}_2) = \text{Sing}(\mathbb{E}_1) \cap \text{Sing}(\mathbb{E}_2)$  and  $\text{ord}_x(\mathbb{E}_1 \cap \mathbb{E}_2) = \min\{\text{ord}_x(\mathbb{E}_1), \text{ord}_x(\mathbb{E}_2)\}$ .

The following basic properties of pairs hold:

**Lemma 1.6.** Let  $\mathbb{E} = (J, b)$  and  $\mathbb{E}_i = (J_i, b_i)$ ,  $i \in \{1, 2, 3, 4\}$ , be pairs on  $Z$ .

- (i) For every  $a \in \mathbb{Z}_+$  we have  $(J^a, ab) \sim (J, b)$ .
- (ii) Let  $m \in \mathbb{Z}_+$  with  $b_1 \mid m$  and  $b_2 \mid m$ . Then

$$(J_1, b_1) \cap (J_2, b_2) \sim \left( J_1^{\frac{m}{b_1}} + J_2^{\frac{m}{b_2}}, m \right).$$

- (iii) We always have  $(J_1 J_2, b_1 + b_2) \supset (J_1, b_1) \cap (J_2, b_2)$ . If further  $\text{Sing}(J_i, b_i + 1) = \emptyset$  for  $i \in \{1, 2\}$ , then the previous inclusion becomes an equivalence.
- (iv) If  $\mathbb{E}_1 \subset \mathbb{E}_2$  and  $\mathbb{E}_3 \subset \mathbb{E}_4$ , then  $\mathbb{E}_1 \cap \mathbb{E}_3 \subset \mathbb{E}_2 \cap \mathbb{E}_4$ . In particular  $\mathbb{E}_1 \sim \mathbb{E}_2$  implies by symmetry  $\mathbb{E}_1 \cap \mathbb{E}_3 \sim \mathbb{E}_2 \cap \mathbb{E}_3$ .
- (v) Let  $\pi : Z' \rightarrow Z$  be a permissible blow-up for  $\mathbb{E}_1$  and  $\mathbb{E}_2$ . Then  $(\mathbb{E}_1 \cap \mathbb{E}_2)' \sim \mathbb{E}_1' \cap \mathbb{E}_2'$ .

*Proof.* All claims follow by looking at the definitions. For a more detailed proof see Lemma 1.1.8 in [Sc1], where we only have to replace the base field  $k$  by  $\mathbb{Z}$ .  $\square$

By (i) we may extend the definition of pairs  $(J, b)$  to  $b \in \mathbb{Q}_+$ : Suppose  $b = \frac{c}{d} \in \mathbb{Q}_+$ , where the greatest common divisor of  $c, d \in \mathbb{Z}_+$  is 1. Then we define  $(J, b)$  to be a pair with assigned number  $b \in \mathbb{Q}_+$  which is equivalent to  $(J^d, b d)$ .

The following is an example, where the assumptions of the second part of (iii) do not hold. For a strategy for constructing such examples see [Sc1] Remark 1.1.9.

**Example 1.7.** Consider the ideals  $J_1 = \langle x^3 + y^5 \rangle$ ,  $J_2 = \langle xz^2 + y^3 \rangle \subset \mathbb{Z}[x, y, z]$ . Since the blow-up with center  $V(x, y)$  is permissible for  $(J_1 J_2, 4)$  but not for  $(J_2, 2)$ , we have  $(J_1 J_2, 4) \not\subset (J_1, 2) \cap (J_2, 2)$ .

Another important result is the following

**Proposition 1.8** (Numerical Exponent Theorem; [H5], Theorem 5.1). *Let  $\mathbb{E}_1 = (J_1, b_1)$  and  $\mathbb{E}_2 = (J_2, b_2)$  be pairs on  $Z$ . If  $\mathbb{E}_1 \subset \mathbb{E}_2$ , then*

$$\text{ord}_x(\mathbb{E}_1) \leq \text{ord}_x(\mathbb{E}_2) \quad \text{for all } x \in Z.$$

*By symmetry  $\mathbb{E}_1 \sim \mathbb{E}_2$  implies  $\text{ord}_x(\mathbb{E}_1) = \text{ord}_x(\mathbb{E}_2)$  for all  $x \in Z$ . In particular we get  $\text{Sing}(\mathbb{E}_1) = \text{Sing}(\mathbb{E}_2)$  if  $\mathbb{E}_1 \sim \mathbb{E}_2$ .*

The last statement implies that  $\text{Sing}(\mathbb{E}_\sim)$  is an invariant of the idealistic exponent.

*Proof.* Consider the local situation at  $x \in Z$ . We introduce a new variable  $t$  and construct a LSB  $S(\alpha, \beta)$  in the following way: First blow up  $\alpha$ -times the origin, where we consider each time the  $T$ -chart. After this we blow up  $\beta$ -times with center  $V(t)$ .

Suppose there exists an  $x_0 \in Z$  with  $\text{ord}_{x_0}(\mathbb{E}_1) > \text{ord}_{x_0}(\mathbb{E}_2)$ . Set  $\alpha_0 := b_1 b_2$  and  $\beta_0 = (\text{ord}_{x_0}(\mathbb{E}_1) - 1)\alpha_0$ . Then  $S(\alpha_0, \beta_0)$  is permissible for  $\mathbb{E}_1$ , but not for  $\mathbb{E}_2$ . This contradicts  $\mathbb{E}_1 \subset \mathbb{E}_2$  and the claim follows.

For a more detailed proof see Theorem 5.1 in [H5] or Proposition 1.1.10 in [Sc1].  $\square$

The converse of the Numerical Exponent theorem is in general false. More precisely the condition  $\text{ord}_x(\mathbb{E}_1) \leq \text{ord}_x(\mathbb{E}_2)$  for all  $x \in Z$  is not stable under permissible blow-ups. An easy example is given by  $\mathbb{E}_1 = (y^2 + x^3, 2)$  and  $\mathbb{E}_2 = (x^2 + y^3, 2)$  over  $\mathbb{A}_{\mathbb{Z}}^2$ .

**Notation.** Let  $m \in \mathbb{N}_0$  be a non-negative integer and  $Z$  as usual a regular scheme (resp. let  $R$  be a regular ring). Then we denote by  $\text{Diff}_{\mathbb{Z}}^{\leq m}(Z)$  (resp.  $\text{Diff}_{\mathbb{Z}}^{\leq m}(R)$ ) the (absolute) differential operators of  $\mathcal{O}_Z$  (resp.  $R$ ) on itself.

**Proposition 1.9** (Diff Theorem; [H5], Theorem 3.4). *Let  $\mathbb{E} = (J, b)$  be a pair on  $Z$  and  $m \in \mathbb{N}_0$ . Let  $\mathcal{D}$  be a left  $\mathcal{O}_Z$ -submodule of  $\text{Diff}_{\mathbb{Z}}^{\leq m}(Z)$ . Then*

$$(J, b) \subset (\mathcal{D}J, b - m)$$

*or equivalently  $(J, b) \sim (\mathcal{D}J, b - m) \cap (J, b)$ .*

If  $m \geq b$ , then the assigned number of  $(\mathcal{D}J, b - m)$  is not positive and hence is a priori not defined. But the singular locus  $\{x \in Z \mid \text{ord}_x(\mathcal{D}J) \geq b - m\}$  is still defined. Then  $\text{Sing}(\mathcal{D}J, b - m) = Z$  and the claim follows immediately.

We also use this proposition in the case of a single differential operator  $\mathcal{D} \in \text{Diff}_{\mathbb{Z}}^{\leq m}(Z)$ ; here we identify  $\mathcal{D}$  with the submodule of  $\text{Diff}_{\mathbb{Z}}^{\leq m}(Z)$  generated by  $\mathcal{D}$ .

*Proof.* Since the situation does not change by extending  $Z$  to  $Z[t]$ , we may assume  $(t) = \emptyset$ . Let  $D \subset Z$  be an arbitrary closed regular subscheme and let  $\pi : Z' \rightarrow Z$  be the blow-up with center  $D$ . We show:

- (i) If  $\pi$  is permissible for  $(J, b)$ , then so it is for  $(\mathcal{D}J, b - m)$ .

- (ii) The relation between the transforms of  $(J, b)$  and  $(\mathcal{D}J, b - m)$  under  $\pi$  is the same as before.

Let  $y \in Z$  be a generic point of  $D$ . The first part follows by the fact that  $J_y \subset M^l$ , for some  $l \in \mathbb{Z}_+$ , implies  $\mathcal{D}J_y \subset M^{l-m}$ . (For example see [H5], Lemma 3.1).

For (ii) we have to show that there exists an  $\mathcal{O}_{Z'}$ -submodule  $\mathcal{D}'$  of  $\text{Diff}_{\mathbb{Z}}^{\leq m}(Z')$  such that  $(\mathcal{D}J)' = \mathcal{D}'J'$ , with  $(\mathcal{D}J, b - m)' = ((\mathcal{D}J)', b - m)$  and  $(J, b)' = (J', b)$ .

*Caution:* Do not forget that the transformations are given by different laws, namely  $(\mathcal{D}J)\mathcal{O}_{Z'} = H^{b-m}(\mathcal{D}J)'$  and  $J\mathcal{O}_{Z'} = H^bJ'$ , where  $H \subset \mathcal{O}_{Z'}$  denotes the ideal sheaf of the exceptional divisor.

Let  $Q \subset \mathcal{O}_Z$  be the ideal sheaf corresponding to the center  $D$ . Then the required property holds for  $\mathcal{D}' := H^{-b+m} \cdot \mathcal{D} \cdot Q^b$  (viewed as an  $\mathcal{O}_{Z'}$ -left module in the function field of  $Z$ ). It is only left to verify  $\mathcal{D}' \subset \text{Diff}_{\mathbb{Z}}^{\leq m}(Z')$ .

The last part and more details are given in the proof of Theorem 3.4 in [H5] or Proposition 1.1.13 in [Sc1].  $\square$

Let  $(f_1, \dots, f_m)$  denote a set of generators of the ideal  $J$  and let  $\mathcal{D}$  be as before. Instead of  $\mathcal{D}J$  we want to apply the Diff Theorem for the ideal generated by  $(\mathcal{D}f_1, \dots, \mathcal{D}f_m)$ . In general, these two ideals do not coincide. We frequently use the Diff Theorem in the following slightly modified version:

**Corollary 1.10.** *Let  $\mathbb{E} = (J, b)$  on  $Z$  and  $\mathcal{D} \subset \text{Diff}_{\mathbb{Z}}^{\leq m}(Z)$  be as in the previous theorem. Further let  $(f_1, \dots, f_m)$  be a set of generators of the ideal  $J$ . Then*

$$(J, b) \subset (\langle \mathcal{D}f_1, \dots, \mathcal{D}f_m \rangle, b - m)$$

or equivalently  $(J, b) \sim (\langle \mathcal{D}f_1, \dots, \mathcal{D}f_m \rangle, b - m) \cap (J, b)$ .

*Proof.* By Proposition 1.9 we have  $(J, b) \subset (\mathcal{D}J, b - m)$ . Further  $\langle \mathcal{D}f_1, \dots, \mathcal{D}f_m \rangle \subseteq \mathcal{D}J$  implies  $(\mathcal{D}J, b - m) \subset (\langle \mathcal{D}f_1, \dots, \mathcal{D}f_m \rangle, b - m)$  (use Lemma 1.6(ii)). This shows the assertion.  $\square$

Since this is an immediate consequence of the Diff Theorem, we do not distinguish between the corollary and the proposition. If we use them, then we refer only to the Diff Theorem, Proposition 1.9.

## 2. TANGENT CONE, DIRECTRIX AND RIDGE

Let  $x \in Z$  be an arbitrary point and let  $R = \mathcal{O}_{Z,x}$  be the regular local ring with maximal ideal  $M$  and residue field  $K = R/M$ . Therefore we can associate the tangent space of  $Z$  at  $x$

$$T_x(Z) := \text{Spec}(gr_x(Z)),$$

where  $gr_x(Z) = \bigoplus_{a \geq 0} M^a / M^{a+1}$ .

Let further  $\mathbb{E} = (J, b)$  be a pair on  $Z$ . By abuse of notation we neglect in  $\mathbb{E}_x = (J_x, b)$  the index  $x$  and write also  $\mathbb{E} = (J, b)$ . In the following we introduce the tangent cone, the directrix and the ridge of  $\mathbb{E}$  at  $x$  and we discuss the aspect of their uniqueness up to equivalence. In order to get the last point we give an interpretation of these objects as idealistic exponents.

**Definition 2.1.** Let  $f \in R$  and  $b \in \mathbb{Q}_+$  with  $b \leq \text{ord}_x(f)$ . We define the  $b$ -initial form of  $f$  (with respect to  $M$ ) as

$$in(f, b) := \begin{cases} f \bmod M^{b+1}, & \text{if } b \in \mathbb{Z}_+, \\ 0, & \text{if } b \notin \mathbb{Z}_+. \end{cases}$$

Note that  $b < \text{ord}_x(f)$  implies  $in(f, b) = 0$ .

**Definition 2.2.** Let  $\mathbb{E} = (J, b)$  be a pair on  $Z$  and  $x \in \text{Sing}(\mathbb{E})$ . Then we define the tangent cone  $T_x(\mathbb{E}) \subset T_x(Z)$  of  $\mathbb{E}$  at  $x$  as the subspace generated by the homogeneous ideal  $In_x(J, b) \subset gr_x(Z)$ , where

$$In_x(J, b) := In_x(\mathbb{E}) := \begin{cases} \langle J \bmod M^{b+1} \rangle = \langle in(f, b) \mid f \in J \rangle, & \text{if } b \in \mathbb{Z}_+, \\ \langle 0 \rangle, & \text{if } b \notin \mathbb{Z}_+. \end{cases}$$

Let  $\mathbb{E}_1 = (J_1, b_1), \mathbb{E}_2 = (J_2, b_2)$  be two pairs on  $Z$ . Then we set

$$In_x(\mathbb{E}_1 \cap \mathbb{E}_2) = In_x(\mathbb{E}_1) + In_x(\mathbb{E}_2)$$

**Remark 2.3.** (1) The ideal  $In_x(J, b) \subset gr_x(Z)$  is well-defined and generated by homogeneous elements of degree  $b$ , because  $x \in \text{Sing}(\mathbb{E})$  and thus  $\text{ord}_x(J) \geq b$ .  
(2) The tangent cone  $T_x(\mathbb{E})$  is *not* invariant under the equivalence relation  $\sim$ . We overcome this later by using an idealistic interpretation of the tangent cone.  
(3) If  $\text{ord}_x(\mathbb{E}_2) > 1$ , then  $In_x(\mathbb{E}_2) = \langle 0 \rangle$  and thus  $T_x(\mathbb{E}_1 \cap \mathbb{E}_2) = T_x(\mathbb{E}_1)$ .

Let us for the moment consider a more general situation: Let  $K$  be a field, consider the polynomial ring  $S = K[W] = K[W_1, \dots, W_n]$  as a graded ring and let  $I \subset S$  be a homogeneous ideal. Then  $I$  defines a cone  $C = \text{Spec}(S/I)$ . In this setting we can define the directrix and the ridge of  $C$  which go back to Hironaka and Giraud.

**Definition 2.4.** The directrix  $\text{Dir}(C)$  of the cone  $C$  is the smallest  $K$ -subvectorspace  $T = \bigoplus_{j=1}^r KY_j \subset S_1 = \bigoplus_{i=1}^n KW_i$  generated by elements  $Y_1, \dots, Y_r \in S_1$  (homogeneous of degree one) such that

$$(I \cap K[Y_1, \dots, Y_r])S = I.$$

Hence  $T = \bigoplus_{j=1}^r KY_j$  is the minimal  $K$ -subspace such that  $I$  is generated by elements in  $K[Y_1, \dots, Y_r]$ . (i.e.  $(Y_1, \dots, Y_r)$  is the smallest list of variables to describe the generators of  $I$ ).

We also say  $(Y) = (Y_1, \dots, Y_r)$  defines the directrix and we implicitly assume that  $r$  is minimal. By abuse of notation we denote the vector space in  $\mathbb{A}_K^n = \text{Spec}(S)$  corresponding to  $\text{Dir}(C)$  also by  $\text{Dir}(C)$ . Further we call  $\text{IDir}(C) := \langle Y_1, \dots, Y_r \rangle$  the ideal of the directrix.

Recall that a polynomial  $\varphi \in K[W] = S$  is called additive if for any  $x, y \in K^n$  we have  $\varphi(x+y) = \varphi(x) + \varphi(y)$ .

**Definition 2.5.** The ridge (or *faîte* in French)  $\text{Rid}(C)$  of the cone  $C$  is the smallest additive subspace  $K[\varphi_1, \dots, \varphi_l] \subset S$  generated by additive homogeneous polynomials  $\varphi_1, \dots, \varphi_l \in S$  such that

$$(I \cap K[\varphi_1, \dots, \varphi_l])S = I.$$

As above we say  $(\varphi_1, \dots, \varphi_l)$  defines the ridge, identify  $\text{Rid}(C)$  with the group subscheme which it defines in  $\mathbb{A}_K^n$  and we call  $\text{IRid}(C) := \langle \varphi_1, \dots, \varphi_l \rangle$  the ideal of the ridge.

**Remark 2.6.** In the case of  $\text{char}(K) = 0$  the additive polynomials are those homogeneous of degree one. Thus the previous definitions coincide in this situation,  $\text{Dir}(C) = \text{Rid}(C)$ .

If  $p = \text{char}(K) > 0$  is positive, then the additive homogeneous polynomials are of the form  $\varphi = \sum_{i=1}^n \lambda_i W_i^q$ ,  $\lambda_i \in K$  and  $q = p^e$ ,  $e \in \mathbb{Z}_{\geq 0}$ . If moreover  $K$  is perfect, then  $\varphi = \psi^q$  for some  $\psi \in K[W]$  homogeneous of degree one. Hence the directrix is the reduction of the ridge,  $\text{Dir}(C) = (\text{Rid}(C))_{\text{red}}$ , if  $K$  is perfect.

For arbitrary  $K$  and  $\lambda \in K$ , we do not know if there is an element  $\rho \in K$  such that  $\rho^q = \lambda$ ,  $q = p^e$  as before. But there is a purely inseparable finite extension  $K(\lambda)/K$  such that this property holds in  $K(\lambda)$ ; e.g.  $K(\lambda) = K[X]/\langle X^q - \lambda \rangle$ . Since  $\{\lambda_i^{(j)} \in K \mid \varphi_j = \sum_{i=1}^n \lambda_i^{(j)} W_i^{q_j}, q_j = p^{e_j}, e_j \in \mathbb{Z}_{\geq 0}, j \in \{1, \dots, l\}\}$  is a finite set, there exists a purely inseparable finite extension  $K'/K$  such that  $\text{Dir}(C)_{K'} = (\text{Rid}(C)_{K'})_{\text{red}}$ , where  $(\cdot)_{K'} = (\cdot) \times_K K'$ .

For more details on the ridge (and in particular an intrinsic definition) see [G1] and [BHM].

Coming back to our situation ( $S = gr_x(Z)$ ,  $C = T_x(\mathbb{E}) = \text{Spec}(S/In_x(\mathbb{E}))$ ), we have

**Definition 2.7.** Let  $\mathbb{E} = (J, b)$  be a pair on  $Z$ . Then we define

- (1) the directrix of  $\mathbb{E}$  at  $x$  by  $\text{Dir}_x(\mathbb{E}) := \text{Dir}_x(T_x(\mathbb{E}))$ ,
- (2) and the ridge of  $\mathbb{E}$  at  $x$  by  $\text{Rid}_x(\mathbb{E}) := \text{Rid}_x(T_x(\mathbb{E}))$ .

Let  $\mathbb{E}_1 = (J_1, b_1)$  and  $\mathbb{E}_2 = (J_2, b_2)$  be two pairs on  $Z$ . Then

$$\text{Dir}_x(\mathbb{E}_1 \cap \mathbb{E}_2) = \text{Dir}_x(\mathbb{E}_1) \cap \text{Dir}_x(\mathbb{E}_2) \quad \text{and} \quad \text{Rid}_x(\mathbb{E}_1 \cap \mathbb{E}_2) = \text{Rid}_x(\mathbb{E}_1) \cap \text{Rid}_x(\mathbb{E}_2)$$

We now come to the *idealistic interpretation* of the tangent cone  $T_x(\mathbb{E})$ , the directrix  $\text{Dir}_x(\mathbb{E})$  and the ridge  $\text{Rid}_x(\mathbb{E})$  of  $\mathbb{E}$  at  $x \in \text{Sing } \mathbb{E}$ .

**Observation 2.8.** Before we start, we want to point out the following:

- (1) Let  $\mathbb{E} = (J, b)$  be a pair on  $Z$ . By Lemma 1.6(i)  $\mathbb{E}$  is equivalent to  $\mathbb{E}^a := (J^a, ab)$  for all  $a \in \mathbb{Z}_+$ . Let  $x \in \text{Sing } (\mathbb{E}) = \text{Sing } (\mathbb{E}^a)$  and  $R = \mathcal{O}_{Z,x}$  as before. We denote by  $K$  the residue field of  $R$  and  $(w) = (w_1, \dots, w_n)$  should be a r.s.p. for  $R$ . Then  $gr_x(Z) \cong K[W] = K[W_1, \dots, W_n]$ , where  $W_i$  denotes the image of  $w_i$  in  $M/M^2$  ( $M = \langle w_1, \dots, w_n \rangle$ ), and  $In_x(\mathbb{E}) = \langle in(f, b) \mid f \in J \rangle$ . We want to show the following equality of ideals in  $gr_x(Z)$ :

$$In_x(\mathbb{E}^a) = (In_x(\mathbb{E}))^a.$$

Clearly,  $in(f + g, b) = in(f, b) + in(g, b)$  for  $f, g \in J$ . Consider an element  $g \in J^a$  which is of the form  $g = g_1 \cdots g_a$  for  $g_1, \dots, g_a \in J$ . Since  $x \in \text{Sing } (\mathbb{E})$  the initials  $in(g_i, b)$  are either zero or homogeneous of degree  $b$  ( $\text{ord}_x(g_i) \geq b$ ) for all  $i \in \{1, \dots, a\}$ . Thus  $in(g, ab) = \prod_{i=1}^a in(g_i, b)$  and we get the desired equality  $In_x(\mathbb{E}^a) = (In_x(\mathbb{E}))^a$ .

If we put  $\mathbb{T}_x(\mathbb{E}) := (In_x(\mathbb{E}), b)$  and  $\mathbb{T}_x(\mathbb{E}^a) := (In_x(\mathbb{E}^a), ab)$ , then the last equation implies that these are equivalent pairs on  $T_x(Z) = \text{Spec}(gr_x(Z))$ .

Let  $IDir_x(\mathbb{E}) = \langle Y_1, \dots, Y_r \rangle$  be the ideal of the directrix with elements  $Y_j \in K[W]$ ,  $1 \leq j \leq r$ , which are homogeneous of degree one. By definition of the directrix, the generators of  $In_x(\mathbb{E})$  are contained in  $K[Y_1, \dots, Y_r]$  and  $(Y)$  is minimal with this condition. This implies that the generators of  $In_x(\mathbb{E}^a)$  are contained in  $K[Y_1, \dots, Y_r]$  and  $(Y)$  is also minimal: Suppose this is wrong, say they are contained in  $K[Z_1, \dots, Z_s]$  for some  $s < r$ . Then the same is true for the generators of  $In_x(\mathbb{E})$  which is a contradiction. This shows

$$\text{Dir}_x(\mathbb{E}) = \text{Dir}_x(\mathbb{E}^a).$$

In particular,  $\mathbb{D}ir_x(\mathbb{E}) := (IDir_x(\mathbb{E}), 1)$  and  $\mathbb{D}ir_x(\mathbb{E}^a) := (IDir_x(\mathbb{E}^a), 1)$  are equivalent pairs on  $T_x(Z)$ .

Now let  $IRid_x(\mathbb{E}) = \langle \varphi_1, \dots, \varphi_l \rangle$  be the ideal of the ridge with additive homogeneous polynomials  $\varphi_i \in K[Y_1, \dots, Y_r] \subset K[W]$ ,  $1 \leq i \leq l$ . Since  $\varphi_i$  is additive, the order is some  $p$ -power, say  $p^{d_i}$  for some  $d_i \in \mathbb{N}_0$  ( $p = \text{char}(K)$ ). Let  $p^c$  ( $c \in \mathbb{N}_0$ ) be the maximal  $p$ -power dividing  $a$ . Then

$$IRid_x(\mathbb{E}^a) = \langle \varphi_1^{p^c}, \dots, \varphi_l^{p^c} \rangle.$$

Hence  $\mathbb{R}id_x(\mathbb{E}) = \bigcap_{i=1}^l (\varphi_i, p^{d_i})$  and  $\mathbb{R}id_x(\mathbb{E}^a) = \bigcap_{i=1}^l (\varphi_i^{p^c}, p^{d_i+c})$  are equivalent pairs on  $T_x(Z)$ .

- (2) Let  $\mathbb{E}_1 = (J_1, b)$  and  $\mathbb{E}_2 = (J_2, b)$  be two pairs on  $Z$  and further  $x \in \text{Sing } (\mathbb{E}_1 \cap \mathbb{E}_2)$ . By definition  $In_x(\mathbb{E}_1 \cap \mathbb{E}_2) = In_x(\mathbb{E}_1) + In_x(\mathbb{E}_2)$  and this is equal to

$$In_x(J_1, b) + In_x(J_2, b) = In_x(J_1 + J_2, b).$$

Hence  $T_x((J_1, b) \cap (J_2, b)) = T_x(J_1 + J_2, b)$  and this implies the equality of the corresponding directrices and ridges.

This observation gives the hint that the tangent cone (resp. the ridge) of equivalent pairs might be related if we use an idealistic interpretation. Hence we introduce the following definitions of the tangent cone, the directrix and the ridge as pairs and prove that these actually give well-defined idealistic exponents. The variant of the tangent cone appeared already in the language of idealistic filtrations ([K], Definition 1.1.2) and also in the language of Rees algebras ([BeV], section 1.2), but the concepts of directrix and ridge considered as pairs are completely new.



**Definition 2.9.** Let  $\mathbb{E} = (J, b)$  be a pair on  $Z$  and  $x \in \text{Sing}(\mathbb{E})$ . Recall that  $K$  denotes the residue field of  $Z$  at  $x$  and  $p = \text{char}(K) \geq 0$ . Let further  $\text{IDir}_x(\mathbb{E}) = \langle Y_1, \dots, Y_r \rangle$  and  $\text{IRid}_x(\mathbb{E}) = \langle \varphi_1, \dots, \varphi_l \rangle$  for elements  $Y_j$  homogeneous of degree one,  $1 \leq j \leq r$ , and additive homogeneous polynomials  $\varphi_i$  of order  $p^{d_i}$ ,  $1 \leq i \leq l$ . Then we define the following pairs on  $T_x(Z) = \text{Spec}(gr_x(Z))$ :

$$\mathbb{T}_x(\mathbb{E}) = (In_x(\mathbb{E}), b) \quad (\text{idealistic tangent cone of } \mathbb{E} \text{ at } x),$$

$$\text{Dir}_x(\mathbb{E}) = (\text{IDir}_x(\mathbb{E}), 1) \quad (\text{idealistic directrix of } \mathbb{E} \text{ at } x),$$

$$\text{Rid}_x(\mathbb{E}) = \bigcap_{i=1}^l (\varphi_i, p^{d_i}) \quad (\text{idealistic ridge of } \mathbb{E} \text{ at } x).$$

If we have two pairs on  $Z$ , say  $\mathbb{E}_1 = (J_1, b_1)$  and  $\mathbb{E}_2 = (J_2, b_2)$ , and  $x \in \text{Sing}(\mathbb{E}_1 \cap \mathbb{E}_2)$ , then

$$\mathbb{T}_x(\mathbb{E}_1 \cap \mathbb{E}_2) = \mathbb{T}_x(\mathbb{E}_1) \cap \mathbb{T}_x(\mathbb{E}_2) = (In_x(\mathbb{E}_1), b_1) \cap (In_x(\mathbb{E}_2), b_2),$$

$$\text{Dir}_x(\mathbb{E}_1 \cap \mathbb{E}_2) = \text{Dir}_x(\mathbb{E}_1) \cap \text{Dir}_x(\mathbb{E}_2) = (\text{IDir}_x(\mathbb{E}_1) + \text{IDir}_x(\mathbb{E}_2), 1),$$

$$\text{Rid}_x(\mathbb{E}_1 \cap \mathbb{E}_2) = \text{Rid}_x(\mathbb{E}_1) \cap \text{Rid}_x(\mathbb{E}_2).$$

**Remark 2.10.** By Observation 2.8 we have for an arbitrary pair and a positive integer  $a \in \mathbb{Z}_+$

$$\mathbb{T}_x(J, b) \sim \mathbb{T}_x(J^a, ab), \quad \text{Dir}_x(J, b) = \text{Dir}_x(J^a, ab), \quad \text{Rid}_x(J, b) \sim \text{Rid}_x(J^a, ab).$$

Further we have seen in the observation that for two pairs with the same assigned number  $\mathbb{T}_x(J_1, b) \cap \mathbb{T}_x(J_2, b) = \mathbb{T}_x(J_1 + J_2, b)$ , which implies the equalities of the corresponding idealistic directrices and ridges.

**Lemma 2.11.** Let  $\mathbb{E} = (J, b)$  be a pair on  $Z$  and  $x \in \text{Sing}(\mathbb{E})$ . Then we have

- (i)  $\text{Dir}_x(\mathbb{E}) \subset \text{Rid}_x(\mathbb{E}) \subset \mathbb{T}_x(\mathbb{E})$ .
- (ii)  $\text{Dir}_x(\mathbb{E}) = \text{Sing}(\text{Dir}_x(\mathbb{E})) \subseteq \text{Sing}(\text{Rid}_x(\mathbb{E})) \subseteq \text{Sing}(\mathbb{T}_x(\mathbb{E})) \subseteq T_x(Z)$ .

*Proof.* Let  $(Y) = (Y_1, \dots, Y_r)$  be the elements (homogeneous of degree one) which determine  $\text{Dir}_x(\mathbb{E})$  and extend these by  $(U) = (U_1, \dots, U_e)$  such that  $gr_x(Z) = K[U, Y]$ . Further let  $(\varphi) = (\varphi_1, \dots, \varphi_l)$  be the additive homogeneous polynomials which yield  $\text{Rid}_x(\mathbb{E})$ .

Since the generators of all three pairs are homogeneous, the extension of the base by  $K[T_1, \dots, T_a]$  does not change the situation. Hence, we may consider the pairs on  $K[U, Y]$ . Since by definition the generators of  $In_x(\mathbb{E})$  are contained in  $K[\varphi] \subset K[Y]$ , any center which is permissible for  $\text{Dir}_x(\mathbb{E})$  (resp.  $\text{Rid}_x(\mathbb{E})$ ) is so for  $\text{Rid}_x(\mathbb{E})$  (resp.  $\mathbb{T}_x(\mathbb{E})$ ). After blowing up either  $\text{Dir}_x(\mathbb{E})$  (resp.  $\text{Rid}_x(\mathbb{E})$ ) is resolved or the situation is still the same. This shows (i).

The first equality and the last inclusion of (ii) follow by definition and (i) implies the rest via the Numerical Exponent Theorem, Proposition 1.8.  $\square$

In characteristic zero or if the characteristic  $p > 0$  is greater than  $b$ , we have the following

**Corollary 2.12.** Let  $\mathbb{E} = (J, b)$  be a pair on  $Z$  and  $x \in \text{Sing}(\mathbb{E})$ . Assume  $\text{char}(K) = 0$  or  $b < \text{char}(K)$ , where  $K$  denotes the residue field of  $Z$  at  $x$ . Then

$$\text{Dir}_x(\mathbb{E}) \sim \text{Rid}_x(\mathbb{E}) \sim \mathbb{T}_x(\mathbb{E}).$$

In particular  $\text{Dir}_x(\mathbb{E}) = \text{Sing}(\text{Dir}_x(\mathbb{E})) = \text{Sing}(\text{Rid}_x(\mathbb{E})) = \text{Sing}(\mathbb{T}_x(\mathbb{E}))$ .

*Proof.* By Lemma 2.11 we only have to show  $\mathbb{T}_x(\mathbb{E}) \subset \text{Dir}_x(\mathbb{E})$ . Let  $(R = \mathcal{O}_{Z,x}, M, K)$  be the local ring of  $Z$  at  $x$  and let  $(u, y) = (u_1, \dots, u_e, y_1, \dots, y_r)$  be a r.s.p. for  $R$  such that  $\text{IDir}_x(\mathbb{E}) = \langle Y_1, \dots, Y_r \rangle$ , where  $Y_j$  denotes the image of  $y_j$  in  $M/M^2$ . Then

$$\text{Dir}_x(\mathbb{E}) = (\langle Y_1, \dots, Y_r \rangle, 1) \sim (Y_1, 1) \cap \dots \cap (Y_r, 1).$$

Recall that  $\mathbb{T}_x(\mathbb{E}) = (In_x(\mathbb{E}), b)$ . By definition of the directrix, the generators of  $In_x(\mathbb{E})$  are contained in  $K[Y]$  and each  $Y_j$  appears to a non-zero power. Hence they lie in  $\langle Y \rangle^b \setminus \langle Y \rangle^{b+1}$  and every generator  $F \in In_x(\mathbb{E})$  can be written as

$$F = \sum_{\substack{B \in \mathbb{Z}_{\geq 0}^r \\ |B|=b}} C_B Y^B,$$

for some  $C_B \in K$ . Further for every  $1 \leq j \leq r$  there exists a generator  $F(j) \in In_x(\mathbb{E})$  such that there is a  $B(j) = (B_1, \dots, B_r) \in \mathbb{Z}_{\geq 0}^r$  with  $C_{B(j)} \neq 0$  and  $B_j \geq 1$ . Therefore this is an

element of  $In_x(\mathbb{E})$ , where  $Y_j$  appears. Set  $M(j) := B(j) - e_j \in \mathbb{Z}_{\geq 0}^r$ . (Here  $e_j \in \mathbb{Z}_{\geq 0}^r$  denotes the  $j$ -th unit vector with zero everywhere except the  $j$ -th place, there is a one). Note that  $|M(j)| = b - 1$ . Let  $\mathcal{D}_{M(j)} \in \text{Diff}_K^{\leq b-1}(K[Y])$  be the differential operator which is defined via

$$\mathcal{D}_{M(j)}(C Y^B) = \binom{B}{M(j)} C Y^{B-M(j)}.$$

for  $C \in K$  and  $B \in \mathbb{Z}_{\geq 0}^r$ . In particular,  $\mathcal{D}_{M(j)}(C Y^{B(j)}) = C \binom{B(j)}{M(j)} Y_j = C B_j Y_j$  and  $\mathcal{D}_{M(j)}(C Y^B) = 0$  if  $|B| = b$  and  $B \notin M(j) + \mathbb{Z}_+^r$ ; consequently

$$\mathcal{D}_{M(j)}(F(j)) = C_{B(j)} B_j Y_j + \sum_{B'(i)} C_{B'(i)} B'_i Y_i,$$

where  $B'(i) = (B'_1, \dots, B'_r) \in \{M(j) + e_i \mid i \in \{1, \dots, r\} \setminus \{j\}\}$ . Since we have  $1 \leq B_j \leq b$  and  $\text{char}(K) = 0$  or  $b < \text{char}(K)$ , we get that  $B_j$  (and thus  $C_{B(j)} B_j$ ) is a unit in  $K$ . We set

$$Y_j^* := (C_{B(j)} B_j)^{-1} \mathcal{D}_{M(j)} F(j) = Y_j + \sum_{B'(i)} (C_{B(j)} B_j)^{-1} C_{B'(i)} B'_i Y_i \in K[Y].$$

Let  $j = 1$ . We choose in  $R$  a system of representatives of  $K = R/M$  and define with this  $y_1^* \in R$  by replacing  $(Y)$  by  $(y)$  in the definition of  $Y_1^*$ . The system  $(y_1^*, y_2, \dots, y_r)$  fulfills the same properties as  $(y)$ . So we may consider the r.s.p.  $(u; y_1^*, y_2, \dots, y_r)$  instead of  $(u, y)$  and put  $\mathcal{D}_1 := \mathcal{D}_{M(1)}$ . (Note that  $\mathcal{D}_1$  is defined in terms of  $(Y)$ ). Then we repeat the above procedure to obtain  $y_2^*$  and  $\mathcal{D}_2$ . After that we determine  $y_3^*$  and  $\mathcal{D}_3 \dots$  We continue until we get  $(y^*) = (y_1^*, \dots, y_r^*)$ . Then the Diff Theorem 1.9 yields for all  $j \in \{1, \dots, r\}$  that  $(F(j), b) \subset (\mathcal{D}_j F(j), 1) = (Y_j^*, 1)$ . This implies

$$\mathbb{T}_x(\mathbb{E}) = (In_x(\mathbb{E}), b) \subset (Y_1^*, 1) \cap \dots \cap (Y_r^*, 1) = \mathbb{D}ir_x(\mathbb{E}).$$

□

**Remark 2.13.** For the arbitrary case the equivalences need not hold. One reason is that  $B_j$  may be zero in  $K$ . Therefore the assumption  $\text{char}(K) = 0$  or  $b < \text{char}(K)$  is essential. In fact, the equivalence  $\mathbb{R}id_x(\mathbb{E}) \sim \mathbb{T}_x(\mathbb{E})$  is even true without the assumption on the characteristic. A detailed proof of this result will appear in [DSc].

If  $K$  is perfect, then  $\mathbb{D}ir_x(\mathbb{E}) = (Y_1, 1) \cap \dots \cap (Y_r, 1) \sim (Y_1^{p^{d_1}}, p^{d_1}) \cap \dots \cap (Y_r^{p^{d_r}}, p^{d_r}) = \mathbb{R}id_x(\mathbb{E})$ , for certain  $d_j \in \mathbb{Z}_{\geq 0}$ .

We have seen that there is not necessarily a relation between the tangent cones  $T_x(\mathbb{E})$  of equivalent pairs. For idealistic interpretations we have the following strong result.

**Proposition 2.14** (Proposition A). *Let  $\mathbb{E}_1 = (J_1, b_1)$  and  $\mathbb{E}_2 = (J_2, b_2)$  be two pairs on  $Z$  with  $\mathbb{E}_1 \subset \mathbb{E}_2$  and  $x \in \text{Sing}(\mathbb{E}_1) \subseteq \text{Sing}(\mathbb{E}_2)$ . Then we have*

- (i)  $\mathbb{T}_x(\mathbb{E}_1) \subset \mathbb{T}_x(\mathbb{E}_2)$ .
- (ii)  $\mathbb{D}ir_x(\mathbb{E}_1) \subseteq \mathbb{D}ir_x(\mathbb{E}_2)$  and hence  $\mathbb{D}ir_x(\mathbb{E}_1) \subset \mathbb{D}ir_x(\mathbb{E}_2)$ .
- (iii)  $\mathbb{R}id_x(\mathbb{E}_1) \subset \mathbb{R}id_x(\mathbb{E}_2)$ .

By symmetry we get equivalence  $\sim$  and equality if  $\mathbb{E}_1 \sim \mathbb{E}_2$ .

This yields that the idealistic version of the tangent cone, the directrix and the ridge are uniquely determined by  $x$  and the equivalence class of  $\mathbb{E}$ . i.e. by the idealistic exponent  $\mathbb{E}_\sim$ . Thus  $\mathbb{E}_\sim$  yields well-defined idealistic exponents  $\mathbb{T}_x(\mathbb{E})_\sim$ ,  $\mathbb{D}ir_x(\mathbb{E})_\sim$ , and  $\mathbb{R}id_x(\mathbb{E})_\sim$ .

In the special situation over perfect fields the implication  $\mathbb{E}_1 \sim \mathbb{E}_2 \Rightarrow \mathbb{D}ir_x(\mathbb{E}_1) = \mathbb{D}ir_x(\mathbb{E}_2)$  was already proven in [H2], Proposition 19.2

*Proof.* By Observation 2.8 the result holds in the case  $\mathbb{E} = (J, b) \sim (J^a, ab)$  for some  $a \in \mathbb{Z}_+$ . Hence it suffices to consider  $b_1 = b_2 = b$ . As always we denote  $R = \mathcal{O}_{Z,x}$  with maximal ideal  $M$  and residue field  $K$ . Let  $(w) = (w_1, \dots, w_n)$  be a r.s.p. for  $R$ .

Let  $\mathbb{E} = (J, b) \in \{(\mathbb{E}_1)_x, (\mathbb{E}_2)_x\}$ . First we extend the base  $R$  to  $R_0 = R \times_K K[t]$ , where  $t$  is an independent indeterminate. Then  $(w, t)$  is a r.s.p. for  $R_0$ . We use the notation  $\mathbb{E}_0 = (J^{(0)} = J[t], b)$  and  $V_0 = \text{Spec}(R_0)$ . Let  $L_0 \subset V_0$  denote the line  $V(w)$  and

$x_0 \in L_0 \subset Z_0$  the origin  $V(w, t)$ . We now consider for  $\alpha \in \mathbb{Z}_+$  the following LSB, which is permissible for  $\mathbb{E}$  (since  $x \in \text{Sing}(\mathbb{E})$ ),

$$\begin{array}{ccccccc}
 L_0 & \cong & L_1 & \cong & \dots & \cong & L_{\alpha-1} & \cong & L_\alpha \\
 \cap & & \cap & & & & \cap & & \cap \\
 (2.1) \quad V_0 & \xleftarrow{\pi_1} & Z_1 \supset V_1 & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_{\alpha-1}} & Z_{\alpha-1} \supset V_{\alpha-1} & \xleftarrow{\pi_\alpha} & Z_\alpha \supset V_\alpha \\
 \Psi & & \Psi & & & & \Psi & & \Psi \\
 x_0 & & x_1 & & \dots & & x_{\alpha-1} & & x_\alpha,
 \end{array}$$

where  $\pi_i : Z_i \rightarrow V_{i-1}$  is the blow-up with center  $x_{i-1} \in V_{i-1}$ ,  $L_i \cong L_0$  is the strict transform of  $L_0$ ,  $x_i$  denotes the unique intersection point of  $L_i$  with the exceptional divisor of the blow-up  $\pi_i$  and  $V_i = \text{Spec}(R_i) \subset Z_i$  is the  $T$ -chart of the blow-up,  $i \in \{1, \dots, \alpha\}$ . Recall that  $L_0 = V(w)$ , hence  $x_i = V(\frac{w}{t^i}, t)$ ,  $L_i = V(\frac{w}{t^i})$  and  $(\frac{w}{t^i}, t)$  is a r.s.p. for  $R_i$ .

Let  $f \in J^{(0)}$  be an arbitrary element. In the  $M$ -adic completion of  $R$  we have  $f(w) = f_0(w) + h(w)$ , where  $f_0$  denotes the part homogeneous of degree  $b$  and  $h(w) \in \langle w \rangle^{b+1}$ . Let  $d = d(f) = \text{ord}_x(h)$ ,  $d > b$ . The transform of  $f$  in  $V_\alpha$  is given by

$$(2.2) \quad f^{(\alpha)}\left(\frac{w}{t^\alpha}, t\right) = f_0^{(\alpha)}\left(\frac{w}{t^\alpha}\right) + t^{\alpha \cdot (d-b)} \cdot h_*, \quad \text{for some } h_* \in \left\langle \frac{w}{t^\alpha} \right\rangle^{b+1} + \langle t \rangle.$$

Recall that  $x_\alpha = V(\frac{w}{t^\alpha}, t)$ . It is clear that the generators of the ideal of the tangent cone (and thus also its idealistic version) at  $x$  did not change under the extension of the base and by the previous we see that the tangent cone at  $x_\alpha$  is the same as the one before the permissible LSB (2.1); just replace in  $\text{In}_x(J^{(0)}, b)$  the coordinates  $(w)$  by  $(\frac{w}{t^\alpha})$  in order to get  $\text{In}_{x_\alpha}(J^{(\alpha)}, b)$ .

Given  $\mathbb{E}_1 \subset \mathbb{E}_2$ , then we can perform the above permissible LSB and get  $\mathbb{E}_1^{(\alpha)} \subset \mathbb{E}_2^{(\alpha)}$  on  $V_\alpha$ . Further every  $f^{(\alpha)} \in J_1^{(\alpha)}$  and  $g^{(\alpha)} \in J_2^{(\alpha)}$  is of the form (2.2). Now choose  $\alpha$  so large that

$$\alpha \cdot (d(f) - b) \geq b \quad \text{and} \quad \alpha \cdot (d(g) - b) \geq b \quad (*)$$

for every  $f^{(\alpha)} \in J_1^{(\alpha)}$  and  $g^{(\alpha)} \in J_2^{(\alpha)}$ .

For simplicity let us drop the indices and assume from the very beginning that  $\mathbb{E}_1 \subset \mathbb{E}_2$  on  $V_0$  are of the special type described above. By the previous discussion this is justified. As usual capital letters  $(W, T)$  denote the images of  $(w, t)$  in  $\langle w, t \rangle / \langle w, t \rangle^2$ . We want to point out that by (2.2) the generators of  $\text{In}_x(\mathbb{E}_1)$  and  $\text{In}_x(\mathbb{E}_2)$  are contained in  $K[W]$ . Hence we consider  $\mathbb{T}_x(\mathbb{E}_1)$  and  $\mathbb{T}_x(\mathbb{E}_2)$  as pairs on  $\text{Spec}(K[W])$ .

Since the tangent cones are generated by homogeneous elements, an extension by some independent indeterminates  $(t') = (t'_1, \dots, t'_a)$  for some  $a \in \mathbb{Z}_+$  does not affect the situation. So it suffices to consider the case without an extension of the base.

For (i) we first want to show

$$(2.3) \quad \mathbb{T}_x(\mathbb{E}_1) \subset \mathbb{T}_x(\mathbb{E}_2).$$

Suppose this is wrong. Then there exists a LSB  $(\diamond)$  over  $K[W]$  which is permissible for  $\mathbb{T}_x(\mathbb{E}_1)$ , but not for  $\mathbb{T}_x(\mathbb{E}_2)$ . By (2.2)  $\text{In}_x(\mathbb{E}_1)$  is generated by the  $f_0(W)$  and  $\text{In}_x(\mathbb{E}_2)$  by the  $g_0(W)$  (for  $f \in J_1$  and  $g \in J_2$ ). We can lift the centers of  $(\diamond)$  back to  $R$  (by choosing a system of representatives of  $K = R/M$  in  $R$  and using  $(w)$  instead of  $(W)$ ). Further we can intersect them with  $V(t)$  and obtain a LSB over  $R_0$ . Because of the special form (2.2) we get by blowing up these modified centers a LSB  $(\hat{\diamond})$  over  $V_0$ , which is permissible for  $\mathbb{E}_1$  by the permissibility of  $(\diamond)$  and property  $(*)$  of  $\alpha$ . But since  $(\diamond)$  is not permissible for  $\mathbb{T}_x(\mathbb{E}_2) = (\text{In}_x(\mathbb{E}_2), b)$ , we also have that  $(\hat{\diamond})$  is *not* permissible for  $\mathbb{E}_2$ . This contradicts  $\mathbb{E}_1 \subset \mathbb{E}_2$  and proves (i).

Now we come to (ii),  $\text{Dir}_x(\mathbb{E}_1) \subseteq \text{Dir}_x(\mathbb{E}_2)$ . By Lemma 2.11  $\text{Dir}_x(\mathbb{E}_i) \subseteq \text{Sing}(\mathbb{T}_x(\mathbb{E}_i))$  and by definition of the directrix it is a permissible center for  $\mathbb{T}_x(\mathbb{E}_i)$ ,  $i \in \{1, 2\}$ . Further (2.3) implies that  $\text{Dir}_x(\mathbb{E}_1)$  is a permissible center for  $\mathbb{T}_x(\mathbb{E}_2)$ , which contains the origin. By the minimality of the directrix  $\text{Dir}_x(\mathbb{E}_2)$  any permissible center containing the origin must lie in  $\text{Dir}_x(\mathbb{E}_2)$ . This implies  $\text{Dir}_x(\mathbb{E}_1) \subseteq \text{Dir}_x(\mathbb{E}_2)$ . The second part of (ii) is clear.

Part (iii),  $\text{Rid}_x(\mathbb{E}_1) \subset \text{Rid}_x(\mathbb{E}_2)$ , is similar to (i). Assume  $\text{Rid}_x(\mathbb{E}_1) \not\subset \text{Rid}_x(\mathbb{E}_2)$ , then there exists a LSB over  $K[W]$  which is permissible for  $\text{Rid}_x(\mathbb{E}_1)$ , but not for  $\text{Rid}_x(\mathbb{E}_2)$ . By the definition of the ridge, this LSB is permissible for  $\text{T}_x(\mathbb{E}_1)$ , but not for  $\text{T}_x(\mathbb{E}_2)$ . This is a contradiction to (2.3). (Alternatively one could lift the LSB as in the proof of (i) to one over  $R_0$  and this yields a contradiction to  $\mathbb{E}_1 \subset \mathbb{E}_2$  as before).  $\square$

### 3. IDEALISTIC COEFFICIENT EXPONENTS AND MAXIMAL CONTACT

An important tool to study the singularities at a point  $x \in Z$  in characteristic zero is the coefficient ideal with respect to a closed subscheme of maximal contact.

We now give the precise definition of the coefficient ideal in the idealistic setting. But we do not restrict our attention to characteristic zero and admit an arbitrary residue field of  $Z$  at  $x$ . It is known that the concept of maximal contact does not work in full generality, therefore we define the coefficient pair with respect to any regular subvariety  $W = V(z) = V(z_1, \dots, z_n)$  containing  $x$ ; we only want to assume that  $(z)$  is part of a r.s.p. for the local ring  $R$  of  $Z$  at  $x$ . (The interesting case for us is, when  $W = V(y_1, \dots, y_s)$  ( $s \leq r$ ), where  $(y) = (y_1, \dots, y_r)$  is such that the image of  $(y)$  in  $gr_x(Z)$  defines the directrix  $\text{Dir}_x(\mathbb{E})$ ).

**Definition 3.1.** Let  $\mathbb{E} = (J, b)$  be a pair on  $Z$  and  $x \in Z$ . Let  $(R = \mathcal{O}_{Z,x}, M, K)$  be the regular local ring of  $Z$  at  $x$ . We consider a fixed system of elements  $(u) = (u_1, \dots, u_d)$  which can be extended to a r.s.p. for  $R$ . Let  $(z) = (z_1, \dots, z_s)$  be elements of  $R$  such that  $(u, z)$  is a r.s.p. for  $R$ . We define the coefficient pair  $\mathbb{D}_x(\mathbb{E}, u, z)$  of  $\mathbb{E}$  at  $x$  with respect to  $(z)$  as the pair on  $W = \text{Spec}(K[[u]])$  which is given by the following construction: Any  $f \in J_x$  may be written (in the  $M$ -adic completion  $\hat{R}$ ) as

$$f = f(u, z) = \sum_{B \in \mathbb{Z}_{\geq 0}^s} f_B(u) z^B$$

with  $f_B(u) \in K[[u]]$ . Then we set  $\mathbb{D}(f, u, z) := \bigcap_{\substack{B \in \mathbb{Z}_{\geq 0}^s \\ |B| < b}} (f_B(u), b - |B|)$  and define further

$$\mathbb{D}_x(\mathbb{E}, u, z) := \bigcap_{f \in J_x} \mathbb{D}(f, u, z) = \bigcap_{l=0}^{b-1} (I(l, u, z), b - l),$$

where  $I(l, u, z) = \langle f_B \mid f \in J_x, B \in \mathbb{Z}_{\geq 0}^s : |B| = l \rangle$ .

The idea of coefficient ideals goes back to Hironaka (in the context of idealistic exponents this appears in [H4] Theorem 1.3, p.908, and [H2] section 8, Theorem 5, p.111) and was developed by Villamayor (for basic objects) and Bierstone-Milman (for presentations).

We may consider  $\mathbb{E}_x$  and  $\mathbb{D}_x(\mathbb{E}, u, z)$  as pairs on  $\hat{R}$ . Then we have  $\mathbb{E}_x \subset \mathbb{D}_x(\mathbb{E}, u, z)$  by construction.

In our context one of the first questions coming into one's mind may be the following: Are the coefficient pairs of equivalent pairs also equivalent? For the idealistic approach there is no reference known to the author where this is proven. Hence we give the affirmative answer in

**Theorem 3.2.** *Let  $\mathbb{E}_1 \subset \mathbb{E}_2$  be two pairs on  $Z$ ,  $x \in Z$ , and  $(u, z) = (u_1, \dots, u_d; z_1, \dots, z_s)$  a r.s.p. for  $(R = \mathcal{O}_{Z,x}, M, K)$ . Then we have*

$$\mathbb{D}_x(\mathbb{E}_1, u, z) \subset \mathbb{D}_x(\mathbb{E}_2, u, z).$$

*By symmetry,  $\mathbb{E}_1 \sim \mathbb{E}_2$  implies  $\mathbb{D}_x(\mathbb{E}_1, u, z) \sim \mathbb{D}_x(\mathbb{E}_2, u, z)$ .*

This implies that an idealistic exponent  $\mathbb{E}_\sim$  determines a unique idealistic exponent  $\mathbb{D}_x(\mathbb{E}, u, z)_\sim$ , called the *idealistic coefficient exponent*.

*Proof.* Let  $\mathbb{E} = (J, b) \in \{\mathbb{E}_1, \mathbb{E}_2\}$ . We consider  $\mathbb{E}_x = (J_x, b)$  on  $R$ . In order to simplify the notation we suppress the index  $x$  and write  $J = J_x$  and  $\mathbb{E} = \mathbb{E}_x$ . First of all let us mention the following easy observation:

Consider  $(J, b) \cap (z, 1)$ . By Lemma 1.6(ii) we may then assume that in the expansion of all  $g \in J$ ,  $g = \sum_B g_B(u) z^B$ , we have  $g_B(u) = 0$  for all  $B \in \mathbb{Z}_{\geq 0}^s$  with  $|B| \geq b$ . (\*)

For  $M \in \mathbb{Z}_{\geq 0}^s$  let  $\mathcal{D}_M \in \text{Diff}_K^{\leq j}(\widehat{R})$ ,  $j = |M|$ , be the differential operator defined by

$$\mathcal{D}_M(C_{A,B} u^A z^B) = \binom{B}{M} C_{A,B} u^A z^{B-M}$$

for  $C_{A,B} \in K$ . In particular  $\mathcal{D}_M(C_{A,M} u^A z^M) = C_{A,M} u^A$ . Further, we define for  $j \in \{1, \dots, b-1\}$  the finite sets

$$S(j) = \{M \in \mathbb{Z}_{\geq 0}^s \mid |M| = j\}.$$

Let  $M \in S(b-1)$ . By the Diff Theorem 1.9 we have  $(J, b) \sim (J, b) \cap (\mathcal{D}_M J, 1)$  and hence

$$(z, 1) \cap (J, b) \sim (z, 1) \cap (J, b) \cap (\mathcal{D}_M J, 1).$$

In here, we have for  $g = \sum_{B \in \mathbb{Z}_{\geq 0}^s} g_B(u) z^B \in J$  (where the expansion is considered in  $\widehat{R}$ )

$$\mathcal{D}_M(g) = g_M(u) + \sum_{\substack{B \in M + \mathbb{Z}_{\geq 0}^s \\ |B| > |M| = b-1}} \binom{B}{M} g_B(u) z^{B-M} = g_M(u),$$

where the last equality follows by (\*). If we apply this to all  $M \in S(b-1)$  and all  $g \in J$ , we get

$$(3.1) \quad (z, 1) \cap \mathbb{E} \sim (z, 1) \cap (J, b) \cap \bigcap_{M \in S(b-1)} (\mathcal{D}_M J, 1) \sim (z, 1) \cap \mathbb{E} \cap \mathbb{D}^{(1)}(\mathbb{E}),$$

where we define  $\mathbb{D}^{(1)}(\mathbb{E}) := (I^{(1)}, 1)$  with

$$I^{(1)} := \langle g_M \mid M \in S(b-1), g \in J \rangle \subset K[[u]].$$

(Note that  $\mathbb{D}^{(1)}(\mathbb{E})$  is a pair on  $\widehat{R}' := K[[u]]$ ). The ideal  $I^{(1)}$  is generated by those  $g_M(u)$  which appear in expansions of elements  $g$  of  $J$  in front of some power  $z^M$  with  $|M| = b-1$ .

By Lemma 1.6 (i) and (iii),

$$(I^{(1)}, 1) \cap (\langle z \rangle^{b-1}, b-1) \sim (I^{(1)}, 1) \cap (z, 1) \cap (\langle z \rangle^{b-1} I^{(1)}, b).$$

Let us consider  $(z, 1) \cap (J, b) \cap \mathbb{D}^{(1)}(\mathbb{E})$ . By part (ii) of Lemma 1.6 we may assume that in the expansion of all  $g \in J$ ,  $g = \sum_B g_B(u) z^B$ , we have  $g_B(u) = 0$  for all  $B \in \mathbb{Z}_{\geq 0}^s$  with  $|B| = b-1$ . Together with (\*) we get  $g_B(u) = 0$  for all  $B \in \mathbb{Z}_{\geq 0}^s$  with  $|B| \geq b-1$ . (\*\*)

Now let  $M \in S(b-2)$ . The Diff Theorem 1.9 yields  $(J, b) \sim (J, b) \cap (\mathcal{D}_M J, 2)$  and therefore

$$(z, 1) \cap (J, b) \cap \mathbb{D}^{(1)}(\mathbb{E}) \sim (z, 1) \cap (J, b) \cap (\mathcal{D}_M J, 2) \cap \mathbb{D}^{(1)}(\mathbb{E}).$$

In here, we have for  $g = \sum_{B \in \mathbb{Z}_{\geq 0}^s} g_B(u) z^B \in J$

$$\mathcal{D}_M(g) = g_M(u) + \sum_{\substack{B \in M + \mathbb{Z}_{\geq 0}^s \\ |B| > |M| = b-2}} \binom{B}{M} g_B(u) z^{B-M} = g_M(u),$$

where the last equality follows by (\*\*). If we apply this to all  $M \in S(b-2)$  and all  $g \in J$ , we get

$$(3.2) \quad (z, 1) \cap \mathbb{E} \cap \mathbb{D}^{(1)}(\mathbb{E}) \sim (z, 1) \cap \mathbb{E} \cap \mathbb{D}^{(1)}(\mathbb{E}) \cap \mathbb{D}^{(2)}(\mathbb{E}),$$

where we define

$$\mathbb{D}^{(2)}(\mathbb{E}) := (I^{(2)}, 2) = \bigcap_{M \in S(b-2)} (\mathcal{D}_M J, 2)$$

with  $I^{(2)} := \langle g_M \mid M \in S(b-2), g \in J \rangle \subset K[[u]]$ . (Again  $\mathbb{D}^{(2)}(\mathbb{E})$  is a pair on  $\widehat{R}' = K[[u]]$ ). Putting (3.1) and (3.2) together gives

$$(3.3) \quad (z, 1) \cap \mathbb{E} \sim (z, 1) \cap \mathbb{E} \cap \mathbb{D}^{(1)}(\mathbb{E}) \cap \mathbb{D}^{(2)}(\mathbb{E}).$$

We go on with this procedure and get at the end

$$(3.4) \quad (z, 1) \cap \mathbb{E} \sim (z, 1) \cap \mathbb{E} \cap \bigcap_{l=1}^{b-1} \mathbb{D}^{(l)}(\mathbb{E}) \sim (z, 1) \cap \bigcap_{l=1}^b \mathbb{D}^{(l)}(\mathbb{E}),$$

where for  $l \in \{1, \dots, b-1\}$  we have  $\mathbb{D}^{(l)}(\mathbb{E}) = (I^{(l)}, l)$  and

$$I^{(l)} = \langle g_M \mid M \in S(b-l), g = \sum g_M(u) y^M \in J \rangle \subset K[[u]].$$

By extending (\*\*) we may assume that in the expansion of an element  $g \in J$ ,  $g = \sum_M g_M(u) z^M$ , we have  $g_M = 0$  for all  $M \in \mathbb{Z}_{\geq 0}^s$  with  $|M| \geq 1$ . So we set

$$I^{(b)} := \langle g_{(0, \dots, 0)} \mid g = \sum g_M(u) z^M \in J \rangle \subset K[[u]]$$

and  $\mathbb{D}^{(b)}(\mathbb{E}) := (I^{(b)}, b)$ .

By construction  $\bigcap_{l=1}^b \mathbb{D}^{(l)}(\mathbb{E}) = \mathbb{D}_x(\mathbb{E}, u, z)$  is a pair on  $K[[u]]$  and therefore does not involve any element of  $(z)$ .

Hence we get for  $\mathbb{E}_1$  and  $\mathbb{E}_2$  (recall  $\mathbb{E}_1 \subset \mathbb{E}_2$ )

$$(3.5) \quad (z, 1) \cap \mathbb{D}_x(\mathbb{E}_1, u, z) \sim (z, 1) \cap \mathbb{E}_1 \subset (z, 1) \cap \mathbb{E}_2 \sim (z, 1) \cap \mathbb{D}_x(\mathbb{E}_2, u, z).$$

Since  $\mathbb{D}_x(\mathbb{E}_1, u, z)$  and  $\mathbb{D}_x(\mathbb{E}_2, u, z)$  are pairs on  $K[[u]]$ , this already implies

$$\mathbb{D}_x(\mathbb{E}_1, u, z) \subset \mathbb{D}_x(\mathbb{E}_2, u, z),$$

which proves the theorem.  $\square$

**Corollary 3.3.** *We want to point out, that (3.4) implies*

$$(z, 1) \cap \mathbb{E} \sim (z, 1) \cap \mathbb{D}_x(\mathbb{E}, u, z)$$

(Keep in mind that we have here the local situation at a point  $x$ ).

By the last theorem,  $\mathbb{D}_x(\mathbb{E}, u, z)$  is invariant under the equivalence relation  $\sim$  if we fix  $(u, z)$ . But we might also consider various choices for  $(z)$ . In this case we have

**Proposition 3.4.** *Let  $\mathbb{E}$  be a pair on  $Z$  and  $x \in Z$ . Fix a system of elements  $(u) = (u_1, \dots, u_d)$  which can be extended to a r.s.p. for  $(R = \mathcal{O}_{Z,x}, M, K)$ . Let  $(z) = (z_1, \dots, z_s)$  and  $(y) = (y_1, \dots, y_s)$  be two possible extensions of  $(u)$ . Assume  $(z, 1) \cap \mathbb{E} \subset (y, 1) \cap \mathbb{E}$ . Then*

$$\mathbb{D}_x(\mathbb{E}, u, z) \subset \mathbb{D}_x(\mathbb{E}, u, y).$$

*By symmetry,  $(z, 1) \cap \mathbb{E} \sim (y, 1) \cap \mathbb{E}$  implies  $\mathbb{D}_x(\mathbb{E}, u, z) \sim \mathbb{D}_x(\mathbb{E}, u, y)$ .*

*Proof.* First of all, Corollary 3.3 and the assumption imply

$$(z, 1) \cap \mathbb{D}_x(\mathbb{E}, u, z) \sim (z, 1) \cap \mathbb{E} \subset (y, 1) \cap \mathbb{E} \sim (y, 1) \cap \mathbb{D}_x(\mathbb{E}, u, y). \quad (*)$$

Let  $(\diamond)$  be a LSB over  $K[[u]]$  which is permissible for  $\mathbb{D}_x(\mathbb{E}, u, z)$ . We can lift it to a LSB  $(\widehat{\diamond})$  over  $K[[u]][z]$  just by intersecting the centers with  $V(z)$ . Then  $(\widehat{\diamond})$  is permissible for  $(z, 1) \cap \mathbb{D}_x(\mathbb{E}, u, z)$  and by  $(*)$  it is so for  $(y, 1) \cap \mathbb{D}_x(\mathbb{E}, u, y)$ . In particular, it is permissible for  $\mathbb{D}_x(\mathbb{E}, u, y)$  and since the latter lives on  $K[[u]]$ , the LSB  $(\diamond)$  is permissible for  $\mathbb{D}_x(\mathbb{E}, u, y)$ . This shows the assertion.  $\square$

Therefore under the special assumption  $(z, 1) \cap \mathbb{E} \sim (y, 1) \cap \mathbb{E}$  the coefficient pair for a fixed system  $(u)$  does not depend on the choice of  $(z)$ . In particular this holds,

- if  $(z, 1) \sim (y, 1)$  or
- if  $(y, 1) \cap \mathbb{E} \sim \mathbb{E} \sim (z, 1) \cap \mathbb{E}$ .

Theorem 3.2 and Proposition 3.4 imply Proposition B.

Let us now introduce the concept of maximal contact, which is an important tool in the proof of resolution of singularities in characteristic zero. Classical references for this are [G2] and [AHV].

**Definition 3.5.** Let  $\mathbb{E} = (J, b)$  be a pair on  $Z$  and  $x \in \text{Sing}(\mathbb{E})$ . Let  $(z) = (z_1, \dots, z_s)$  be a system of elements in the local ring  $(R = \mathcal{O}_{Z,x}, M)$  which can be extended to a r.s.p. for  $R$ . We say  $W := V(z)$  has maximal contact with  $\mathbb{E}$  at  $x$  if the following equivalence holds

$$\mathbb{E}_x = (J_x, b) \sim (z, 1) \cap (J_x, b).$$

In particular, the images of  $(z)$  in  $M/M^2$  are part of a minimal generating system for the directrix  $\text{Dir}_x(\mathbb{E})$ .

We have the following result on the existence of maximal contact

**Lemma 3.6.** Let  $\mathbb{E} = (J, b)$  be a pair on  $Z$ ,  $x \in \text{Sing}(\mathbb{E})$ , and  $(u, y) = (u_1, \dots, u_e, y_1, \dots, y_r)$  be a r.s.p. for  $(R = \mathcal{O}_{Z,x}, M, K)$  such that the images of  $(y)$  in  $M/M^2$  define the  $\text{Dir}_x(\mathbb{E})$ . Assume  $\text{char}(K) = 0$  or  $b < \text{char}(K)$ .

Then there exists a system  $(z) = (z_1, \dots, z_r)$  of elements in  $\widehat{R}$  such that we have for every  $j \in \{1, \dots, r\}$ :

- (i) The images of  $z_j$  and  $y_j$  in  $M/M^2$  coincide.
- (ii) If we set  $(\tilde{u}^{(j)}) := (u, z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_r)$ , then

$$\mathbb{E}_x \sim (z_j, 1) \cap \mathbb{D}_x(\mathbb{E}, \tilde{u}^{(j)}, z_j).$$

In particular,  $\mathbb{E}_x \sim (z, 1) \cap \mathbb{D}_x(\mathbb{E}, u, z)$ , i.e. each  $V(z_j)$  (and thus  $V(z_1, \dots, z_r)$ ) has maximal contact with  $\mathbb{E}$  at  $x$ .

- (iii) There exist  $\mathcal{D}_j \in \text{Diff}_K^{\leq b-1}(K[Y])$  and  $F(j) \in \text{In}_x(\mathbb{E})$  such that  $\mathcal{D}_j(F(j)) = \epsilon_j Z_j$  for some units  $\epsilon_j \in R$ . Further there are  $f(j) \in J\widehat{R}$  which map in  $\text{gr}_x(Z)$  to  $F(j)$  and  $(\mathcal{D}'_j(f(j)), 1) \sim (z_j, 1)$ , where  $\mathcal{D}'_j$  denotes the differential operator on  $\widehat{R}$  induced by  $\mathcal{D}_j$ .

*Proof.* In fact, (i) and (ii) follow immediately from the proof of (iii). Thus we focus on this part. Recall the following from the proof of Corollary 2.12:

Every generator  $F \in \langle Y \rangle^b \setminus \langle Y \rangle^{b+1}$  of  $\text{In}_x(\mathbb{E})$  can be written as  $F = \sum_{B \in \mathbb{Z}_{\geq 0}^r: |B|=b} C_B Y^B$  for some  $C_B \in K$ . Further for  $j = 1$  there exists a generator  $F(j)$  of  $\text{In}_x(\mathbb{E})$  such that there is a  $B(j) = (B_1, \dots, B_r) \in \mathbb{Z}_{\geq 0}^r$  with  $C_{B(j)} \neq 0$  and  $B_j \geq 1$  ( $Y_j$  appears). Set  $M(j) := B(j) - e_j \in \mathbb{Z}_{\geq 0}^r$ ,  $|M(j)| = b - 1$ . Let  $\mathcal{D}_j := \mathcal{D}_{M(j)} \in \text{Diff}_K^{\leq b-1}(K[Y])$  the differential operator which is defined via  $\mathcal{D}_{M(j)}(C Y^B) = \binom{B}{M(j)} C Y^{B-M(j)}$ . Consequently

$$\mathcal{D}_{M(j)}(F(j)) = C_{B(j)} B_j Y_j + \sum_{B'(i)} C_{B'(i)} B'_i Y_i,$$

where  $B'(i) = (B'_1, \dots, B'_r) \in \{M(j) + e_i \mid i \in \{1, \dots, r\} \setminus \{j\}\}$ . The assumptions on  $\text{char}(K)$  imply that  $B_j$  (and thus  $C_{B(j)} B_j$ ) is a unit in  $K$ . Set

$$Y_j^* := (C_{B(j)} B_j)^{-1} \mathcal{D}_{M(j)} F(j) = Y_j + \sum_{B'(i)} (C_{B(j)} B_j)^{-1} C_{B'(i)} B'_i Y_i \in K[Y].$$

We choose as system of representatives of  $K = R/M$  in  $R$  and define with this  $y_1^* \in R$  by replacing  $(Y)$  by  $(y)$  in the  $Y_1^*$ . The system  $(y_1^*, y_2, \dots, y_r)$  fulfills the same properties as  $(y)$ . So we may consider the r.s.p.  $(u; y_1^*, y_2, \dots, y_r)$  instead of  $(u, y)$  and put  $\mathcal{D}_1 := \mathcal{D}_{M(1)}$ . Then we repeat the above procedure for  $j = 2$  to obtain  $y_2^*$  and  $\mathcal{D}_2$ . We continue this until we have obtained  $(y^*) = (y_1^*, \dots, y_s^*)$ .

Denote by  $\mathcal{D}'_j$  the differential operator on  $\widehat{R}$  induced by  $\mathcal{D}_j$ ,  $1 \leq j \leq s$ . ( $\mathcal{D}_j$  extends by acting trivially on  $(u)$ ). Further there exist  $f(j) \in J\widehat{R}$ , which are mapped to  $F(j) \in \text{gr}_x(Z)$  and  $\mathcal{D}'_j(f(j)) = \epsilon_j y_j^* + h_j$  for some units  $\epsilon_j \in R$  and elements  $h_j \in \widehat{R}$ , which do not involve  $y_j^*$ . Set for every  $j \in \{1, \dots, s\}$

$$z_j := y_j^* + \epsilon_j^{-1} h_j.$$

Then  $\mathcal{D}_j(F(j)) = \epsilon_j Z_j$ ,  $(\mathcal{D}'_j(f(j)), 1) = (z_j, 1)$  and by the Diff Theorem 1.9 we have  $\mathbb{E}_x \sim \mathbb{E}_x \cap (z_j, 1)$ . Together with Corollary 3.3 we get  $\mathbb{E}_x \sim (z, 1) \cap \mathbb{D}_x(\mathbb{E}, u, z)$ . This proves (i) and (ii) holds by construction of the elements  $(z)$ .  $\square$

**Corollary 3.7.** Fix  $(u)$  as above and let  $(y)$  and  $(z)$  be two extensions of  $(u)$  to a r.s.p. such that  $V(y)$  and  $V(z)$  have maximal contact. Then we have  $\mathbb{D}_x(\mathbb{E}, u, y) \sim \mathbb{D}_x(\mathbb{E}, u, z)$ .

*Proof.* By the previous lemma and Corollary 3.3

$$(y, 1) \cap \mathbb{E}_x \sim (y, 1) \cap \mathbb{D}_x(\mathbb{E}, u, y) \sim \mathbb{E}_x \sim (z, 1) \cap \mathbb{D}_x(\mathbb{E}, u, z) \sim (z, 1) \cap \mathbb{E}_x$$

and Proposition 3.4 implies  $\mathbb{D}_x(\mathbb{E}, u, y) \sim \mathbb{D}_x(\mathbb{E}, u, z)$ .  $\square$

#### 4. FIRST STEPS WITH POLYHEDRA FOR IDEALISTIC EXPONENTS

First let us explain why polyhedra are useful in the context of resolution of singularities. For this we introduce the Newton polyhedron of a pair.

Let  $\mathbb{E} = (J, b)$  be a pair on  $Z$  and  $x \in \text{Sing}(\mathbb{E})$ . Denote as usually by  $(R = \mathcal{O}_{Z,x}, M, K)$  the local ring of  $Z$  at  $x$ . By abuse of notation we skip the index  $x$  and write  $\mathbb{E} = (J, b)$  instead of  $\mathbb{E}_x$ . Fix a system  $(u) = (u_1, \dots, u_e)$  of elements in  $M$  which can be extended to a r.s.p. for  $R$ . We consider various choices of a system  $(y) = (y_1, \dots, y_r)$  such that  $(u, y)$  is a r.s.p. for  $R$ .

Let  $(f) = (f_1, \dots, f_m)$  be a set of generators of  $J$ . In the  $M$ -adic completion of  $R$  we can write each element  $g \in J$  as

$$(4.1) \quad g = \sum_{(A,B) \in \mathbb{Z}_{\geq 0}^n} C_{A,B} u^A y^B$$

with coefficients  $C_{A,B} \in R^\times \cup \{0\}$ . Denote by  $C_{A,B,i}$  the coefficients of the expansion of  $f_i$ ,  $1 \leq i \leq m$ .

**Definition 4.1.** For the given data we introduce the following objects.

- (1) The Newton polyhedron  $\Delta^N(\mathbb{E}, u, y)$  (or  $\Delta_x^N(\mathbb{E}, u, y)$ ) of  $\mathbb{E} = (J, b)$  at  $x$  with respect to  $(u, y)$  is defined to be the smallest closed convex subset of  $\mathbb{R}_{\geq 0}^n$  containing all elements of the set

$$S(f, b; u, y) := \left\{ \frac{(A, B)}{b} + \mathbb{R}_{\geq 0}^n \mid 1 \leq i \leq m \wedge C_{A,B,i} \neq 0 \wedge |B| \leq b \right\}.$$

Let  $\mathbb{E}'$  be another pair on  $Z$  which is singular at  $x$ . Then  $\Delta^N(\mathbb{E} \cap \mathbb{E}', u, y) \subset \mathbb{R}_{\geq 0}^n$  denotes the smallest closed convex subset containing  $\Delta^N(\mathbb{E}, u, y)$  and  $\Delta^N(\mathbb{E}', u, y)$ .

- (2) Using this we define the polyhedron  $\Delta(\mathbb{E}, u, y) = \Delta_x(\mathbb{E}, u, y)$  of  $\mathbb{E} = (J, b)$  at  $x$  with respect to  $(u, y)$  as the Newton polyhedron of the coefficient pair with respect to  $(y)$ ;

$$\Delta(\mathbb{E}, u, y) := \Delta^N(\mathbb{D}_x(\mathbb{E}, u, y), u) \subseteq \mathbb{R}_{\geq 0}^e.$$

Further  $\Delta(\mathbb{E} \cap \mathbb{E}', u, y) \subset \mathbb{R}_{\geq 0}^e$  denotes the smallest closed convex subset containing  $\Delta(\mathbb{E}, u, y)$  and  $\Delta(\mathbb{E}', u, y)$ .

If there is no confusion possible, we just say  $\Delta^N(\mathbb{E}, u, y)$  is the Newton polyhedron of  $\mathbb{E}$  and  $\Delta(\mathbb{E}, u, y)$  is the polyhedron of  $\mathbb{E}$ .

These polyhedra are not necessarily invariant under the equivalence relation  $\sim$ , see Example 4.9. But they are independent of the choice of the generators  $(f) = (f_1, \dots, f_m)$  of  $J$ . We could define  $\Delta^N(\mathbb{E}, u, y)$  to be the smallest closed convex subset of  $\mathbb{R}_{\geq 0}^n$  containing all the elements of the set

$$\tilde{S}(\mathbb{E}, u, y) := \left\{ \frac{(\tilde{A}, \tilde{B})}{b} + \mathbb{R}_{\geq 0}^n \mid \exists g = \sum_{(A,B)} C_{A,B} u^A y^B \in J : C_{\tilde{A}, \tilde{B}} \neq 0 \wedge |\tilde{B}| \leq b \right\}.$$

In fact, denote by  $\Delta(S)$  the polyhedron generated by some set  $S \subset \mathbb{R}_{\geq 0}^n$ . Then we have:

**Lemma 4.2.** *The Newton polyhedron does not depend on the choice of the generating set  $(f) = (f_1, \dots, f_m)$  of  $J$ . More precisely,*

$$\Delta(S(f, b; u, y)) = \Delta(\tilde{S}(\mathbb{E}, u, y)).$$



*Proof.* Since  $f_1, \dots, f_m \in J$ , we get the inclusion  $\Delta(S(f, b; u, y)) \subseteq \Delta(\tilde{S}(\mathbb{E}, u, y))$ . On the other hand, let  $g \in J = \langle f_1, \dots, f_m \rangle$ . Then  $g = \sum_{i=1}^m \lambda_i f_i$  for  $\lambda_i \in R$  and therefore we get that in the expansion of  $g$  every  $(A, B) \in \mathbb{Z}_{\geq 0}^n$  with non-zero coefficient and  $|B| \leq b$  is contained in  $\Delta(S(f, b; u, y))$ . This yields  $\Delta(S(f, b; u, y)) = \Delta(\tilde{S}(\mathbb{E}, u, y))$ .  $\square$

The definition of the coefficient pair implies that  $\Delta(\mathbb{E}, u, y)$  is the smallest convex subset of  $\mathbb{R}_{\geq 0}^n$  containing

$$(4.2) \quad S_*(f, b; u, y) := \left\{ \frac{A}{b - |B|} + \mathbb{R}_{\geq 0}^n \mid 1 \leq i \leq m \wedge C_{A, B, i} \neq 0 \wedge |B| < b \right\}.$$

**Proposition 4.3.** *The polyhedron  $\Delta(\mathbb{E}, u, y)$  associated to a pair  $\mathbb{E} = (J, b) = (\langle f \rangle, b)$  on  $R$  is a certain projection of the corresponding Newton polyhedron  $\Delta^N(\mathbb{E}, u, y)$ .*

*Proof.* This follows immediately by investigating how the projection of a point  $(A, B)$  from  $(0, \dots, 0, 1) \in \mathbb{R}_{\geq 0}^n$  onto  $\mathbb{R}^{n-1} \times \{0\}$  is determined. Applying this several times we obtain the assertion.

For more details see [Sc1], Proposition 2.1.3 and Lemma 2.4.1.  $\square$

**Corollary 4.4.** *The polyhedron  $\Delta(\mathbb{E}, u, y)$  of a pair  $\mathbb{E} = (J, b)$  is independent of the chosen set of generators  $(f) = (f_1, \dots, f_m)$ .*

*Proof.* This is an immediate consequence of Lemma 4.2 and Proposition 4.3.  $\square$

An important invariant of the singularity of  $\mathbb{E}$  at  $x$  is the order of the coefficient pair with respect to a system  $(y)$  which determines  $\text{Dir}_x(\mathbb{E})$ . Using the following definition this can be recovered from the polyhedron  $\Delta(\mathbb{E}; u; y)$ .

**Definition 4.5.** Let  $\Delta \subset \mathbb{R}_{\geq 0}^n$  be any subset. We define

$$\delta(\Delta) := \inf\{|v| = v_1 + \dots + v_n \mid v = (v_1, \dots, v_n) \in \Delta\}.$$

If  $\Delta = \Delta(\mathbb{E}, u, y)$ , then we set  $\delta_x(\Delta(\mathbb{E}, u, y)) := \delta(\Delta(\mathbb{E}, u, y))$ .

As an immediate consequence of Definition 4.1 we get:

**Lemma 4.6.** *Let  $\mathbb{E} = (J, b)$  be a pair on  $Z$ ,  $x \in \text{Sing}(\mathbb{E})$ , and  $(u, y)$  a r.s.p. for the local ring  $\mathcal{O}_{Z, x}$ . Then  $\delta_x(\Delta(\mathbb{E}, u, y))$  coincides with the order of the coefficient pair  $\mathbb{D}_x(\mathbb{E}, u, y)$ .*

Note that we did not make any further assumptions on the system  $(y)$  (e.g. that it yields the directrix of  $\mathbb{E}$ ).

Although the polyhedra  $\Delta(\mathbb{E}, u, y)$  may change under  $\sim$  or under different choices for  $(y)$ , we have that  $\delta_x(\Delta(\mathbb{E}, u, y))$  is an invariant of the idealistic exponent  $\mathbb{E}_\sim$ . More precisely:

**Proposition 4.7.** *Let  $\mathbb{E} = \mathbb{E}_1 \sim \mathbb{E}_2$  be two equivalent pairs on  $Z$  and  $x \in \text{Sing}(\mathbb{E})$ . Let  $(u, y)$  be a r.s.p. for  $\mathcal{O}_{Z, x}$ .*

(1) *Then we have:*

$$\delta_x(\Delta(\mathbb{E}_1, u, y)) = \delta_x(\Delta(\mathbb{E}_2, u, y)).$$

(2) *Let  $(u, z)$  be another choice for the r.s.p. and suppose  $(z, 1) \cap \mathbb{E} \sim (y, 1) \cap \mathbb{E}$ . Then*

$$\delta_x(\Delta(\mathbb{E}, u, y)) = \delta_x(\Delta(\mathbb{E}, u, z)).$$

*This implies in particular that this number is independent of the choice the maximal contact coordinates.*

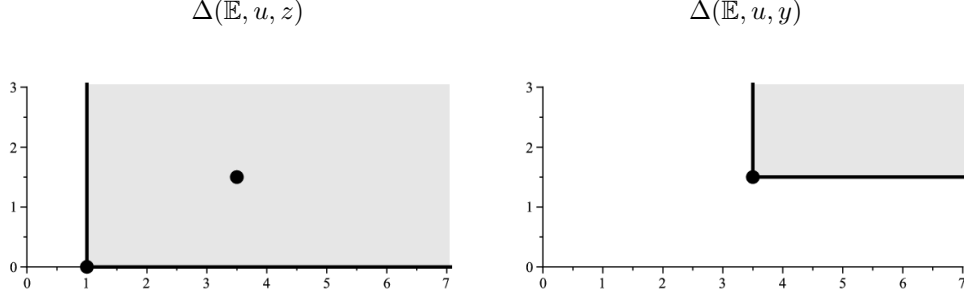
*Proof.* The first (resp. second) part is a consequence of Theorem 3.2 (resp. Proposition 3.4), the Numerical Exponent Theorem, Proposition 1.8, and Lemma 4.6 above.  $\square$

But we want to show something more. In the following two examples we see that  $\delta_x(\Delta(\mathbb{E}, u, y))$  depends on the choice of  $(y)$  and further the polyhedra (and thus the Newton polyhedra) of equivalent pairs may differ. Nevertheless, we want to prove that *for arbitrary characteristic* we are able to maximize  $\delta_x(\Delta(\mathbb{E}, u, y))$  with respect to the choices for  $(y)$ , so that the obtained number depends only on  $\mathbb{E}$ ,  $x$  and  $(u)$ . For this we introduce the intrinsic polyhedron  $\Delta_x(\mathbb{E}, u)$  in section 6.

**Example 4.8.** Consider the pair

$$\mathbb{E} = (y^2 + u_1^7 u_2^3, 2) = (z^2 + 2zu_1^2 + u_1^4 + u_1^7 u_2^3, 2)$$

over any field  $K$ , where  $y := z + u_1^2$  and the point of interest  $x$  is the origin. Then we get  $\delta_x(\Delta(\mathbb{E}, u, y)) = 5$  and  $\delta_x(\Delta(\mathbb{E}, u, z)) = 1$ . The picture looks as follows:



**Example 4.9.** The Newton polyhedron and the polyhedron of  $\mathbb{E}$  may change under the equivalence  $\sim$ . The origin of this example is [BM2], Example 5.14, p.788 and it has been slightly modified and worked out for our setting together with Vincent Cossart.

Let  $K = \mathbb{C}$ ,  $d \in \mathbb{Z}_+$ ,  $d \geq 2$ . We look at the origin of  $\mathbb{A}_{\mathbb{C}}^4$ . Consider the two pairs

$$\mathbb{E}_1 = (z^d - x^{d-1}y^{d-1}, d) \cap (t, 1)$$

$$\mathbb{E}_2 = (z^d - x^{d-1}y^{d-1}, d) \cap (t^{d-1} - x^{d-2}y^{d-1}, d-1)$$

First,  $(t, z)$  yields the directrix in both cases; therefore  $(u) = (x, y)$  and  $(y) = (t, z)$ .

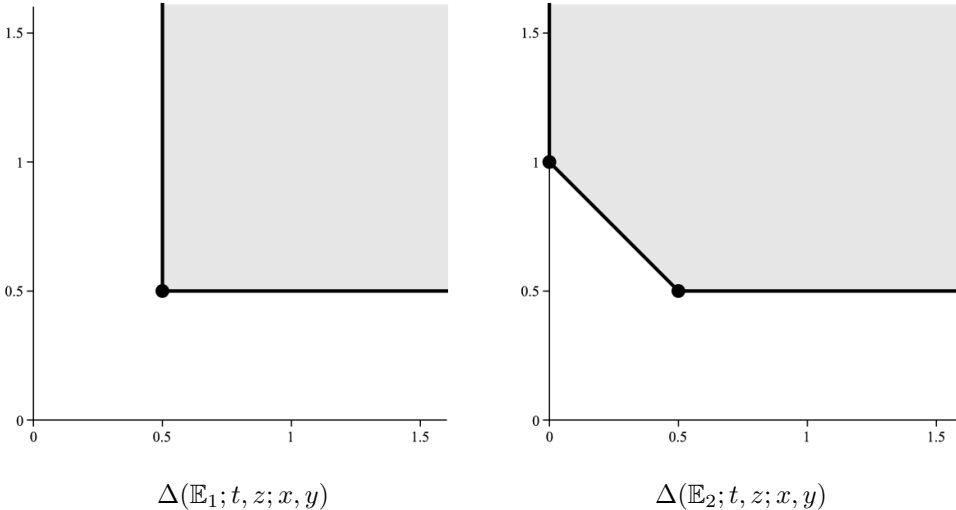
The generating set of the polyhedron associated to  $\mathbb{E}_1$  is  $V_1 = \left\{ \left( \frac{d-1}{d}, \frac{d-1}{d} \right) \right\}$  and the one for  $\mathbb{E}_2$  is  $V_2 = \left\{ \left( \frac{d-1}{d}, \frac{d-1}{d} \right); \left( \frac{d-2}{d-1}, 1 \right) \right\}$ . Clearly the polyhedra are different which implies that also the Newton polyhedra differ.

From the Diff-Theorem, Proposition 1.9, we obtain by applying the differential operators  $\frac{\partial}{\partial x}$  and  $\frac{\partial^{d-2}}{\partial t^{d-2}}$  that

$$\mathbb{E}_1 \sim (z^d - x^{d-1}y^{d-1}, d) \cap (x^{d-2}y^{d-1}, d-1) \cap (t, 1) \sim \mathbb{E}_2.$$

Therefore  $\mathbb{E}_1$  and  $\mathbb{E}_2$  are two equivalent pairs whose associated polyhedra differ!

The picture for  $d = 2$  looks as follows:



The last example plays also a crucial role if there exist exceptional components of a resolution process. It forces us in [Sc2] (or see also [Sc1]) to introduce idealistic exponents with history, which take care of the exceptional components and the preceding resolution process.

## 5. CHARACTERISTIC POLYHEDRA OF AN IDEALISTIC EXPONENT

In this section we define the characteristic polyhedron of a pair by imitating the construction of Hironaka's characteristic polyhedron of a singularity. After that we discuss what the characteristic polyhedron of an idealistic exponent is.

Recall the construction of Hironaka's characteristic polyhedron. More detailed references are section 7 of [CJS], section 2.2 of [Sc1], or Hironaka's original work [H1].

Let  $(R, M, K = R/M)$  be a regular local Noetherian excellent ring,  $J \subset R$  a non-zero ideal and  $(u, y) = (u_1, \dots, u_e; y_1, \dots, y_r)$  a r.s.p. of  $R$ . Note that so far we do not make any other assumptions on  $(u, y)$ , e.g. we do not suppose that  $(y)$  is related to the directrix of  $J$ .

Set  $R' = R/\langle u \rangle$  and  $J' = J \cdot R'$ .

**Definition 5.1.** (1) Let  $f \in R$  be an element in  $R$ ,  $f \notin \langle u \rangle$ . Then we can expand  $f$  in a finite sum

$$(5.1) \quad f = \sum_{(A,B) \in \mathbb{R}_{\geq 0}^{e+r}} C_{A,B} u^A y^B$$

with coefficients  $C_{A,B} \in R^\times \cup \{0\}$ . Denote by  $n = n_{(u)}(f)$  the order of  $f \bmod \langle u \rangle$  in the ideal generated by  $\overline{y_j} = y_j \bmod \langle u \rangle$ ,  $j \in \{1, \dots, r\}$ . Then we define the polyhedron  $\Delta(f; u; y)$  associated to  $(f, u, y)$  as the smallest closed convex subset of  $\mathbb{R}_{\geq 0}^e$  containing all elements of the set

$$\left\{ \frac{A}{n - |B|} + \mathbb{R}_{\geq 0}^e \mid C_{A,B} \neq 0 \wedge |B| < n \right\}.$$

(2) Let  $(f) = (f_1, \dots, f_m)$  be a system of elements in  $R$  with  $f_i \notin \langle u \rangle$  for every  $i$ . Then the polyhedron  $\Delta(f; u; y)$  associated to  $(f, u, y)$  is defined to be the smallest closed convex subset of  $\mathbb{R}_{\geq 0}^e$  containing  $\bigcup_{i=1}^m \Delta(f_i; u; y)$ .

The polyhedron  $\Delta(f; u; y)$  clearly depends on the choice of  $(f) = (f_1, \dots, f_m)$ . A special class of system of generators an ideal  $J$  are so called  $(u)$ -standard bases (see [H1], Definition (2.20)). Since the polyhedra  $\Delta(\mathbb{E}; u; y)$  are independent of the choice of the generators (Corollary 4.4) we do not recall this quite technical definition. We only remark that they are generators  $(f) = (f_1, \dots, f_m)$  of  $J$  such that  $f_i \notin \langle u \rangle$  and which are ordered by the order of  $f \bmod \langle u \rangle$ , and moreover  $m$  is as small as possible.

**Definition 5.2.** Let  $J \subset R$  be a non-zero ideal and  $(u) = (u_1, \dots, u_e)$  a system of elements as before. We define

$$\Delta(J; u) = \bigcap_{(y)} \bigcap_{(f)} \Delta(f; u; y),$$

where the first intersection ranges over all systems  $(y)$  extending  $(u)$  to a r.s.p. of  $R$  and the second runs over all possible  $(u)$ -standard bases  $(f) = (f_1, \dots, f_m)$  of  $J$ . The polyhedron  $\Delta(J; u)$  is called the characteristic polyhedron of  $J$  with respect to  $(u)$ .

This is not Hironaka's original definition. But one can deduce from the following result due to Hironaka that the two definitions coincide.

**Theorem 5.3** ([H1], Theorem (4.8)). *Let  $J \subset R$  be a non-zero ideal and  $(u) = (u_1, \dots, u_e)$  a system of regular elements in  $R$  that can be extended to a r.s.p. of  $R$ . Set  $R' = R/\langle u \rangle$  and  $J' = J \cdot R'$ . Let  $(y) = (y_1, \dots, y_r)$  be a system of elements in  $R$  extending  $(u)$  to a r.s.p. of  $R$  and moreover assume that  $(y)$  yields the ideal generating the directrix of  $J'$ .*

*Then there exists a  $(u)$ -standard basis  $(\hat{f}) = (\hat{f}_1, \dots, \hat{f}_m)$  in  $\hat{R}$  and a system of elements  $(\hat{y}) = (\hat{y}_1, \dots, \hat{y}_r)$  such that  $(u, \hat{y})$  is a r.s.p. of  $\hat{R}$ ,  $(\hat{y})$  determines the directrix of  $J'$  and*

$$\Delta(\hat{f}; u; \hat{y}) = \Delta(J; u).$$

Starting with any r.s.p.  $(u, y)$  and any  $(u)$ -standard basis  $(f)$  Hironaka shows how to obtain  $(\hat{y})$  and  $(\hat{f})$  by applying the procedure of vertex preparation which consists of alternately normalizing the generators and solving the vertices of  $\Delta(f; u; y)$ .

In [CSc1] Cossart and the author extend the result of [CP] and investigate under which conditions it is possible to attain the characteristic polyhedron without going to the completion.

We imitate the definition of  $\Delta(J; u)$  in order to define  $\Delta(\mathbb{E}; u)$ :

**Definition 5.4.** Let  $(J, b)$  be a pair on  $R$  and let  $(u) = (u_1, \dots, u_e)$  be a system of regular elements that can be extended to a r.s.p. of  $R$ . We define

$$\Delta(\mathbb{E}; u) := \bigcap_{(y)} \Delta(\mathbb{E}; u; y),$$

where the intersection ranges over all systems  $(y)$  extending  $(u)$  to a r.s.p. of  $R$  and as before  $\Delta(\mathbb{E}; u; y)$  denotes the polyhedron associated to  $\mathbb{E}$  and  $(u, y)$  (see Definition 4.1 and (4.2)). We call  $\Delta(\mathbb{E}; u)$  the *characteristic polyhedron of the pair  $\mathbb{E}$  with respect to  $(u)$* .

Analogous to Theorem 5.3 we have

**Theorem 5.5** (Theorem C). *Let  $\mathbb{E} = (J, b)$  be a pair on a regular local Noetherian excellent ring  $R$  and denote by  $(u, y) = (u_1, \dots, u_e; y_1, \dots, y_r)$  a r.s.p. for  $R$  such that the initial forms of  $(y)$  yield the whole directrix  $\text{Dir}_x(\mathbb{E})$ .*

*Then there exist elements  $(y^*) = (y_1^*, \dots, y_r^*)$  in  $\hat{R}$  such that  $(u, y^*)$  is a r.s.p. for  $\hat{R}$ ,  $(y^*)$  yields  $\text{Dir}_x(\mathbb{E})$ , and*

$$\Delta(\mathbb{E}; u; y^*) = \Delta(\mathbb{E}; u)$$

*Proof.* We use Hironaka's polyhedron in order to give a different description of  $\Delta(\mathbb{E}; u)$ : Let  $(f) = (f_1, \dots, f_m)$  be a  $(u)$ -standard basis of  $J$ . Let  $(g) := (g_1, \dots, g_l) := (f_{i_1}, \dots, f_{i_l})$ ,  $l \leq m$  and  $1 \leq i_1 < i_2 < \dots < i_l \leq m$ , be those elements of  $(f)$  which fulfill  $n_{(u)}(f_{i_\alpha}) = b$ . Set  $I := \langle g \rangle \subset R$ . Then there is a system  $(x) = (x_1, \dots, x_s) := (y_{j_1}, \dots, y_{j_s})$  with  $s \leq r$  and  $1 \leq j_1 < j_2 < \dots < j_s \leq r$  which is a minimal generating set of its directrix  $\text{Dir}_x(I)$ . Let  $(w) = (w_1, \dots, w_d)$  be the elements  $\{u, y\} \setminus \{x\}$ ,  $d = r + e - s \geq e$ . By definition  $g_{i_\alpha} \notin \langle w \rangle$  for all  $1 \leq \alpha \leq l$  and  $(x)$  also defines the directrix of  $I' = IR'$ , where  $R' = R/\langle w \rangle$ . Hence we can apply Theorem 5.3 and obtain elements  $(\hat{g})$  and  $(\hat{x})$  such that

$$\Delta(I; u) = \Delta(\hat{g}; u; \hat{x}) = \Delta((\hat{g}, b); u; \hat{x}).$$

Let  $(\hat{f}) = (\hat{f}_1, \dots, \hat{f}_m)$  be the  $(u)$ -standard basis which we obtain by putting together  $(\hat{g})$  and the other elements of  $(f)$  which we did not touch. We claim that the associated polyhedron is minimal, i.e.

$$\Delta(\mathbb{E}; u; \hat{x}) = \Delta(\mathbb{E}; u).$$

First of all, the assumption that  $(y)$  yields the whole directrix of  $\mathbb{E}$  implies that  $(g)$  are those elements of  $(f)$  which are of order  $b$  at the origin, and further  $(x) = (y)$  is a minimal generating set of the directrix  $\text{Dir}_x(I)$ .

If we start with another choice for  $(y)$ , say  $(z)$ , then we can apply the above procedure and obtain  $(\hat{z})$  with  $\Delta(\mathbb{E}; u; \hat{z}) \subset \Delta(\mathbb{E}; u; z)$ . Therefore we have to show

$$(5.2) \quad \Delta(\mathbb{E}; u; \hat{y}) = \Delta(\mathbb{E}; u; \hat{z}),$$

which then implies the assertion of the theorem.

By abuse of notation we write in the following  $(y)$  (resp.  $(z)$ ) instead of  $(\hat{y})$  (resp.  $(\hat{z})$ ).

Consider  $h \in J$  and let  $h = \sum_{(A,B)} C_{A,B} u^A y^B$  be an expansion as in (4.1). Then

$$S(h, b; u, y) := \left\{ \frac{A}{b - |B|} \mid C_{A,B} \neq 0 \wedge |B| < b \right\}.$$

contains the vertices of the polyhedron  $\Delta((h, b); u; y)$  and thus generates it. Moreover,  $\Delta(\mathbb{E}; u; y)$  is smallest closed convex subset of  $\mathbb{R}_{\geq 0}^e$  containing  $\bigcup_{h \in J} \Delta((h, b); u; y)$ .

By the assumption we have in  $\widehat{R}$  an expansion of  $y_j$ , for every  $j \in \{1, \dots, r\}$ , which is of the form

$$(5.3) \quad y_j = L_j(z) + H_j(u, z) + Q_j(u), \quad \text{where}$$

◊  $L_j(z) \in K[z]$  are polynomials homogeneous of degree one such that

$$\langle L_1(z), \dots, L_r(z) \rangle = \langle z_1, \dots, z_r \rangle \subset R.$$

◊  $H_j(u, z) \in K[[u, z]]$  are contained in  $\langle u, z \rangle^2$  and  $H_j(u, 0) = 0$ ,

◊  $Q_j(u) \in K[[u]]$  are contained in  $\langle u \rangle^2$ .

We split the substitution from  $(y)$  to  $(z)$  into the following three steps:

$$y_j \xrightarrow{(1)} L_j(z) \xrightarrow{(2)} L_j(z) + H_j(u, z) \xrightarrow{(3)} L_j(z) + H_j(u, z) + Q_j(u).$$

for  $1 \leq j \leq r$ . We show that the polyhedra after each step coincide with  $\Delta(\mathbb{E}; u; y) \subset \mathbb{R}_{\geq 0}^e$ .

In **step (1)**, a monomial  $u^A y^B$  is mapped to  $u^A \prod_{j=1}^r L_j(z)^{B_j}$ . Since every  $L_j(z)$  is homogeneous of degree one, we obtain the same point  $\frac{A}{b-|B|}$  after the substitution. Although some monomials contributing to a point  $v$  of the polyhedron might vanish under the change from  $(y)$  to  $(z)$  it can never happen that all of them disappear, i.e.  $v$  is still appearing in the polyhedron with respect to  $(z)$ .

Next we come to **step (2)**. By the first step, we may assume that it is given by

$$y_j = z_j + H_j(u, z).$$

Consider an expansion of  $H_j(u, z)$  in  $\widehat{R}$ ,  $H_j(u, z) = \sum_{(C,D)} \lambda_{j,C,D} u^C z^D$ , for certain  $\lambda_{j,C,D} \in K = R/M$ . By the assumptions we have  $|D| \geq 1$ , and if  $|D| = 1$  then  $C \neq 0$ . (Otherwise this monomial could already be shifted into  $L_j(z)$ ).

Pick  $(C, D) \in \mathbb{Z}_{\geq 0}^e \times \mathbb{Z}_{\geq 0}^r$  with  $\lambda_{j,C,D} \neq 0$  for some  $j \in \{1, \dots, r\}$ . Let us consider the substitution

$$y_j \xrightarrow{(2_{C,D})} z_j + \lambda_{j,C,D} u^C z^D, \quad 1 \leq j \leq r.$$

An easy computation shows

$$\begin{aligned} y^B &= \sum_{M_1=0}^{B_1} \dots \sum_{M_r=0}^{B_r} \lambda_{M,C,D} (u^C z^D)^{M_1+\dots+M_r} z^{B-(M_1, \dots, M_r)} = \\ &= \sum_{M=0}^B \lambda_{M,C,D} (u^C)^{|M|} z^{B-M+D \cdot |M|}, \end{aligned} \quad (*)$$

where we set  $\lambda_{M,C,D} := \prod_{j=1}^r \binom{B_j}{M_j} \lambda_{j,C,D}^{M_j}$ . Using this for  $C_{A,B} u^A y^B$ , we see that all the points which might appear are of the form

$$v' := \frac{A + C \cdot |M|}{b - (|B - M| + |D| \cdot |M|)}$$

For  $M = (0, \dots, 0)$  we obtain the same point  $\frac{A}{b-|B|}$  back with the same coefficient  $C_{A,B}$ . Since  $|D| \geq 1$ , and  $C \neq 0$  if  $|D| = 1$ , we get

$$v' \in \left( \frac{A}{b-|B|} + \mathbb{R}_{\geq 0}^e \right) \setminus \left\{ \frac{A}{b-|B|} \right\}.$$

Therefore vertices are not touched and the polyhedron does not change under  $(2_{C,D})$ . We apply this for each  $(C, D)$  with non-zero coefficients and get that the polyhedron does not change in step (2).

Finally, we have to deal with **step (3)**: In order to avoid too long notation we set  $\Delta(y) := \Delta((J(u, y), b); u; y) = \Delta((J, b); u; y)$  and  $\Delta(z) := \Delta((J(u, y(u, z)), b); u; z) = \Delta((J, b); u; z)$ . Our goal is to show  $\Delta(y) = \Delta(z)$ . By the first two steps we may assume that the substitution given by

$$y_j = z_j + Q_j(u), \quad 1 \leq j \leq r,$$

for some  $Q_j(u) = \sum_{C \in \mathbb{Z}_{\geq 0}^e} D_{C,j} u^C \in \langle u \rangle^2 \subset K[[u]]$ .

Let  $(\widehat{f}) = (\widehat{f}_1, \dots, \widehat{f}_m)$  be the  $(u)$ -standard base of  $J$  which we obtained by the construction at the beginning of the proof. By abuse of notation we write  $(f) = (f_1, \dots, f_m)$  instead of  $(\widehat{f}) = (\widehat{f}_1, \dots, \widehat{f}_m)$ .

Let  $(f^{(b)}) = (f_1, \dots, f_q)$ ,  $1 \leq q \leq m$ , be those elements of  $(f)$  with  $\text{ord}_x(f_i) = b$  for all  $i \in \{1, \dots, q\}$ . The construction implies the minimality of the polyhedron  $\Delta(f^{(b)}, b; u; y) = \Delta(f^{(b)}; u; y)$  with respect to choices for  $(f^{(b)}; y)$ .

Pick  $C \in \mathbb{Z}_{\geq 0}^e$  with  $D_{C,j} \neq 0$  for some  $j \in \{1, \dots, r\}$ . Let us consider the substitution

$$y_j \stackrel{(3_C)}{=} z_j + D_{C,j} u^C, \quad 1 \leq j \leq r.$$

First of all, this cannot delete any of the vertices of  $\Delta(f^{(b)}; u; y)$  — otherwise we get a contradiction to the minimality of this polyhedron. Further  $(3_C)$  creates the point  $C \in \Delta(f^{(b)}; u; z)$ : Similar to  $(*)$  we have

$$y^B = \sum_{M_1=0}^{B_1} \dots \sum_{M_r=0}^{B_r} D_{M,C} (u^C)^{M_1+\dots+M_r} z^{B-(M_1, \dots, M_r)} = \sum_{M=0}^B D_{M,C} (u^C)^{|M|} z^{B-M}, \quad (*)'$$

where we set  $D_{M,C} := \prod_{j=1}^r \binom{B_j}{M_j} D_{C,j}^{M_j}$ . For  $|B| = b$  we obtain  $C \in \Delta(f^{(b)}; u; z) \subseteq \Delta(f, b; u; z)$ . Here we use the assumption that  $(y)$  yields the *whole* directrix of  $\mathbb{E}$ . If this is not the case, there might not necessarily exist a monomial with  $|B| = b$  such that  $C$  is created in the polyhedron, see Example 5.6 below.

Let us see how  $\Delta(f, b; u; y)$  behaves under the change  $(3_C)$ . By  $(*)'$  we have for arbitrary  $A \in \mathbb{Z}_{\geq 0}^e$  and  $B \in \mathbb{Z}_{\geq 0}^r$

$$C_{A,B} u^A y^B = \sum_{M=0}^B C_{A,B} D_{M,C} u^{C \cdot |M| + A} z^{B-M}.$$

The corresponding points are

$$\begin{aligned} \frac{C \cdot |M| + A}{b - |B| + |M|} &= \frac{|M|}{b - |B| + |M|} \cdot C + \frac{A}{b - |B| + |M|} \\ &= \frac{|M|}{b - |B| + |M|} \cdot C + \frac{b - |B|}{b - |B| + |M|} \cdot \frac{A}{b - |B|} \end{aligned} \quad (**)$$

for  $M = (M_1, \dots, M_r) \in \mathbb{Z}_{\geq 0}^r$  and  $0 \leq M_j \leq B_j$  for all  $j$ . If  $|B| \geq b$ , then  $\frac{|M|}{b - |B| + |M|} \geq 1$  and the first line of  $(**)$  implies  $\frac{C \cdot |M| + A}{b - |B| + |M|} \in C + \mathbb{R}_{\geq 0}^e$ .

So suppose  $|B| < b$ . For  $|M| = 0$  we get  $\frac{A}{b - |B|}$  back and the coefficient of  $u^A z^B$  is  $C_{A,B}$ . The factors before  $\frac{A}{b - |B|}$  and  $C$  in the last line of  $(**)$  are both non-negative, they are smaller or equal one and their sum is

$$\frac{|M|}{b - |B| + |M|} + \frac{b - |B|}{b - |B| + |M|} = 1.$$

Therefore every point  $\frac{C \cdot |M| + A}{b - |B| + |M|}$  is contained in the connecting line between  $\frac{A}{b - |B|}$  and  $C$  (for  $|B| < b$  and  $M \in \mathbb{Z}_{\geq 0}^r$  with  $0 \leq M_j \leq B_j$  for all  $j$ ).

The conclusion is:

- (i) Either  $C \in \Delta(y)$  is already contained in the polyhedron. Then we do not create a new vertex  $C$  under the change  $(3_C)$ . Further we have seen that all points which appear newly are contained on the line between the original point and  $C$  and thus they are in the interior of  $\Delta(y)$ . In particular the vertices are not touched and we get  $\Delta(y) = \Delta(z)$ .
- (ii) Or  $C \notin \Delta(y)$  and  $C$  becomes a vertex of  $\Delta(z)$ . Moreover by the last argument  $\Delta(z)$  is the smallest closed convex subset containing  $C$  and  $\Delta(y)$ .

Together we see that in both cases  $\Delta(y) \subseteq \Delta(z)$ . Up to now we have considered only a part of the substitution  $y_j = z_j + Q_j(u) = z_j + \sum_{C \in \mathbb{Z}_{\geq 0}^e} D_{C,j} u^C$ . We apply this for each  $C$  with non-zero coefficients and get

$$\Delta(y) \subseteq \Delta(z).$$

But the new vertices created in (ii) appear in the polyhedron  $\Delta(f^{(b)}; u; z)$  and can be eliminated. This is a contradiction to the minimality and we obtain the desired equality (5.2):  $\Delta(\mathbb{E}; u; y) = \Delta((f, b); u; y) = \Delta((f, b); u; z) = \Delta(\mathbb{E}; u; z)$ .

This completes the proof of Theorem 5.5.  $\square$

The assumption in Theorem 5.5 that the initial forms of  $(y) = (y_1, \dots, y_r)$  yield the *whole* directrix  $\text{Dir}_x(\mathbb{E})$  is crucial.

**Example 5.6.** Consider the pair  $\mathbb{E} = (\langle f_1, f_2 \rangle, 2)$  over a field  $K$ ,  $\text{char}(K) = p \geq 3$ , given by

$$f_1(u, y) = y_1^2 + h_1(u_1) \quad \text{and} \quad f_2(u, y) = u_3 y_2 + (y_2 + u_2^n)^p + h_2(u_1).$$

for some  $h_1, h_2 \in K[u_1]$  and an integer  $n \in \mathbb{Z}_+$ ,  $n \geq 2$ . The system  $(y_1, y_2, u_3)$  generates the directrix  $\text{Dir}_x(\mathbb{E})$  and the elements with  $n_{(u)}(f_i) = b = 2$  are  $(g) = (f_1)$ . Let  $h_1(u_1)$  be such that  $\Delta(f_1; u; y) = \Delta(f_1; u)$  coincides with the characteristic polyhedron.

Suppose Theorem 5.5 would hold in this case. Then  $\Delta((f, 2); u; y)$  should be independent of the choice of  $(y)$ . For  $(z) = (z_1, z_2) = (y_1, y_2 + u_2^n)$  we get

$$f_1(u, z) = z_1^2 + h_1(u_1) \quad \text{and} \quad f_2(u, z) = u_3 z_2 - u_2^n u_3 + z_2^p + h_2(u_1)$$

and still  $\Delta(f_1; u; z) = \Delta(f_1; u; y) = \Delta(f_1; u)$ . Set  $v := (0, \frac{np}{2}, 0)$  and  $w := (0, \frac{n}{2}, \frac{1}{2})$ . Obviously  $(0, 0, 1), v \in \Delta((f, 2); u; y)$  and  $(0, 0, 1), w \in \Delta((f, 2); u; z)$ . The assumption  $p \neq 2$  implies  $p > 2$  and thus  $\frac{np}{2} > n$ . Therefore  $w \notin \Delta((f, 2); u; y)$  and further  $v \notin \Delta((f, 2); u; z)$ .

The polyhedra  $\Delta((f, 2); u; y)$  and  $\Delta((f, 2); u; z)$  are essentially different.

The previous example illustrates that in general it is not possible to make  $\Delta(f, b; u, y)$  (with our definitions) independent of the choice of the system  $(y)$ . But still we can say something in the previous case, where  $(y)$  does not give the whole directrix. Namely, in both cases of Example 5.6 the point  $(0, 0, 1)$  appears in the polyhedra. Hence  $\delta(\Delta(f, 2; u, y)) = \delta(\Delta(f, 2; u, z)) = 1$ . For the general statement see Lemma 6.4.

Note that for the definition of  $\Delta(J; u)$  the points of the form  $\frac{A}{n_i - |B|}$ ,  $n_i = n_{(u)}(f_i)$ , are considered. On the other hand,  $\Delta(\mathbb{E}; u)$  is defined by those of the form  $\frac{A}{b - |B|}$  and in general  $b \leq n_i$ . Therefore these two polyhedra do not necessarily coincide:

**Example 5.7.** Let  $K$  be a field of characteristic three and set  $b = 2$ . Let

$$f_1 = z_1^2 + u_1^3 \quad \text{and} \quad f_2 = z_2^3 + z_2^2 u_2^2 + u_2^9$$

and  $J = \langle f_1, f_2 \rangle \subset K[u, z]_{\langle u, z \rangle}$ . The polyhedron  $\Delta(f, u, z)$  is generated by  $\left\{ \frac{(3,0)}{2}, \frac{(0,2)}{1}, \frac{(0,9)}{3} \right\}$  and the vertices are  $\text{Vert}(\Delta(f, u, z)) = \{v := (\frac{3}{2}, 0), w := (0, 2)\}$ . One can show that we already have  $\Delta(f, u, z) = \Delta(J, u)$ .

On the other hand,  $\text{Vert}(\Delta(\mathbb{E}; u; z)) = \{v = (\frac{3}{2}, 0), \tilde{v} := (0, \frac{9}{2})\}$ , where we set  $\mathbb{E} := (\langle f_1, f_2 \rangle, b = 2)$ . The vertex  $\tilde{v}$  we can be eliminated by change the coordinates to  $(x_1, x_2) := (z_1, z_2 + u_2^3)$ . Thus  $\Delta(\mathbb{E}; u; z) \neq \Delta(\mathbb{E}; u)$ . Since the order of  $f_2$  at the origin is three and thus bigger than  $b = 2$ , the directrix of  $\mathbb{E} = (\langle f_1, f_2 \rangle, 2)$  is only given by  $Z_1$ ! If we set  $y_1 := z_1$  and  $u_3 := z_2$ , then  $f_2 \in \langle u_1, u_2, u_3 \rangle$ , which means that the assumption  $f_i \notin \langle u \rangle$  of Theorem 5.3 does not hold.

*Therefore there is an essential difference between the polyhedron of the ideal  $J$  and the polyhedron of the pair  $\mathbb{E} = (J, b)$ .*

**Remark 5.8** (*Characteristic polyhedra of idealistic exponents*). The characteristic polyhedron  $\Delta(\mathbb{E}; u)$  may not behave well under the equivalence relation  $\sim$ . Proposition 6.1 implies that in Example 4.9 the polyhedra are already minimal and although the pairs in this example are equivalent, the polyhedra do not coincide.

In Theorem 6.3 below we proof that  $\delta_x(\Delta(\mathbb{E}_1; u)) = \delta_x(\Delta(\mathbb{E}_2; u))$  for two equivalent pairs. Therefore this is an invariant of the idealistic exponent  $\mathbb{E}_\sim$ . Moreover, in [Sc2] this result is used to deduce that the invariant of Bierstone and Milman for their constructive

resolution of singularities in characteristic zero can be obtained solely by considering polyhedra. This means from this point of view it is not necessary to have a unique polyhedron of an idealistic exponent.

One way to get a unique characteristic polyhedron for an idealistic exponent would be to characterize a canonical representant. Since the changes of the polyhedra occur when we apply differential operator a candidate for the canonical representant could be by applying all differential operators  $\text{Diff}_{\mathbb{Z}}(R)$  on  $J$  and then reducing via  $(J^a, ab) \sim (J, b)$ ,  $a \in \mathbb{Z}_+$  as much as possible. In fact, we show in Lemma 6.2 that the reduction in the last step does not change our polyhedron. These things already appear in Hironaka's work [H5], where he uses the notion of *Diff-full* pairs (loc. cit., Definition 11.2) and shows how to obtain such a situation (loc. cit., Lemma 11.2). The previous idea also appears in [BGV], Theorem 3.11, where a canonical representant for a Rees algebra given over a perfect field is detected.

**Remark 5.9** (*Quasi-homogeneous characteristic polyhedra*). Let  $R$  be regular Noetherian local ring and  $(u, y) = (u_1, \dots, u_e; y_1, \dots, y_r)$  be a r.s.p. for  $R$ . So far we never used weights on the elements of  $(u, y)$ , respectively, to be more precise, we assigned to each of them the weight 1. For an element  $g = \sum C_{A,B} u^A y^B \in R$  and a non-negative rational number  $b \in \mathbb{Q}_+$  the polyhedron  $\Delta((g, b); u; y)$  was then defined via the points  $\frac{A}{b - |B|}$  with  $C_{A,B} \neq 0$  and  $|B| < b$  (Definition 4.1 and (4.2)). But in principle we are not forced to consider only this situation – and in order to obtain refined information on the singularity it might also be useful to change the view in certain directions. (Note that the following generalization can also be done for Hironaka's characteristic polyhedron).

Let  $\nu : R \rightarrow \mathbb{Q} \cup \{\infty\}$  be a monomial valuation on  $R$  defined by

$$\nu(u_i) = \alpha_i, \quad \nu(y_j) = \beta_j, \quad \nu(\lambda) = 0, \quad \text{and} \quad \nu(0) = \infty,$$

where  $\lambda \in R^\times$  is a unit in  $R$  and  $\alpha_i, \beta_j \in \mathbb{Q}_{\geq 0}$  are non-negative rational numbers,  $1 \leq i \leq e$  and  $1 \leq j \leq r$ . The example which we are having in mind and which will appear in [Sc2] is, when  $(y)$  determines the directrix of a pair  $\mathbb{E}$ ,  $\Delta(\mathbb{E}; u; y) = \Delta(\mathbb{E}; u)$  is minimal,  $\alpha_i = \frac{1}{\delta} < 1$ , for all  $i$ , where  $\delta := \delta(\Delta(\mathbb{E}; u))$  (Definition 4.5), and  $\beta_j = 1$ , for all  $j$ .

For  $g \in R$  and  $b \in \mathbb{Q}_+$  as above set  $\mathbb{E} = (g, b)$ . Then we define the *associated  $\nu$ -polyhedron*  $\Delta^\nu(\mathbb{E}; u; y)$  as the smallest closed convex subset of  $\mathbb{R}_{\geq 0}^e$  containing all the elements of

$$\left\{ \frac{\alpha \cdot A}{b - |\beta \cdot B|} + \mathbb{R}_{\geq 0}^e \mid C_{A,B} \neq 0 \wedge |\beta \cdot B| < b \right\},$$

where we abbreviate  $\alpha \cdot A := (\alpha_1 A_1, \dots, \alpha_e A_e)$  and  $\beta \cdot B := (\beta_1 B_1, \dots, \beta_r B_r)$ . One possibility to define the *characteristic  $\nu$ -polyhedron* is by

$$\Delta^\nu(\mathbb{E}; u) := \bigcap_{(y)} \Delta^\nu(\mathbb{E}; u; y),$$

where the intersection runs over all systems  $(y)$  extending  $(u)$  to a r.s.p. for  $R$  and which fulfil the additional condition  $\nu(y_j) = \beta_j$ ,  $1 \leq j \leq r$ . We set  $\delta^\nu := \delta(\Delta^\nu(\mathbb{E}; u))$ .

All the notions and results of before can then be developed and proven in this setting. We only remark that in the  $\nu$ -variant of Theorem 5.5 the assumption on the system  $(y)$  becomes:  $(y)$  determines the  $\nu$ -directrix which is the directrix of the  $\nu$ -initial forms on the weighted graded ring, where the weight is induced by  $\nu$ .

Moreover, Theorem 6.3 and Lemma 6.4 below are also true in the quasi-homogeneous situation.

## 6. SOME PROPERTIES

In this final section we state some of the properties of the characteristic polyhedra and on the information they provide.

First, we prove that the polyhedra are independent of the choice of the maximal contact.



**Proposition 6.1.** *Let  $\mathbb{E} = (J, b)$  be a pair on  $(R, M)$ . Fix a system of elements  $(u) = (u_1, \dots, u_d)$  which can be extended to a r.s.p. for  $R$ . Let  $(y) = (y_1, \dots, y_s) \subset R$  such a possible extension and suppose further that  $V(y)$  has maximal contact with  $\mathbb{E}$  at the origin. Then the polyhedron  $\Delta(\mathbb{E}; u; y)$  is independent of the choice of  $(y)$  with these properties. This means if  $(z) \subset R$  is another extension of  $(u)$  and  $V(z)$  has maximal contact, then*

$$\Delta(\mathbb{E}; u; y) = \Delta(\mathbb{E}; u; z).$$

*Proof.* Since both have maximal contact with  $\mathbb{E}$ , we have by definition

$$\mathbb{E} \sim \mathbb{E} \cap (y, 1) \cap (z, 1) \quad (*)$$

Since  $(u, z)$  is a r.s.p. for  $R$ , we can express  $(y)$  by these elements, and as in the proof of Theorem 5.5 (see (5.3)) we have in  $\hat{R}$  an expansion  $y_j = L_j(z) + Q_j(u) + H_j(u, z)$ , where  $L_j(z) \in K[z]$  are polynomials homogeneous of degree one,  $Q_j(u) \in \langle u \rangle^2$ , and  $H_j(u, z) \in \langle u, z \rangle^2$  with  $H_j(u, 0) = 0$ , for any  $j \in \{1, \dots, s\}$ .

Let  $g \in J$  and consider an  $M$ -adic expansion of this element

$$g = \sum_{A, B} C_{A, B} u^A y^B.$$

As we already have seen in the proof of Theorem 5.5 we do not change the polyhedron if we insert  $y_j = L_j(z) + Q_j(u) + H_j(u, z)$ . The vertices are fixed and the points coming from  $Q_j(u)$  appear by  $(*)$  already before the change from  $(y)$  to  $(z)$ . All other points, which may occur, lie on the connecting line between some point of the generating set of  $\Delta(\mathbb{E}; u; y)$  and some point coming from  $Q_j(u)$ .  $\square$

We have seen that the polyhedra may change under the equivalence relation  $\sim$ . But we can also say when the polyhedra is stable.

**Lemma 6.2.** *Let  $\mathbb{E} = (J, b)$  and  $\mathbb{E}_i = (J_i, b_i)$ ,  $i \in \{1, 2\}$ , be pairs on  $Z$  and  $x \in \text{Sing}(\mathbb{E})$ . As usual  $(R, M, K)$  denotes the regular local ring of  $Z$  at  $x$  and  $(t) = (t_1, \dots, t_n) = (u, y)$  is a r.s.p. for  $R$ . We consider the situation at  $x$  and abbreviate the notation by  $\Delta(J, b) := \Delta((J, b); u; y)$ .*

- (i) *If  $a \in \mathbb{Z}_+$ , then  $\Delta(J, b) = \Delta(J^a, ab)$ .*
- (ii) *Suppose  $b_1, b_2 \in \mathbb{Z}_+$  and let  $c \in \mathbb{Z}_+$  with  $b_1 \mid c$  and  $b_2 \mid c$ . Then*

$$\Delta((J_1, b_1) \cap (J_2, b_2)) = \Delta\left(J_1^{\frac{c}{b_1}} + J_2^{\frac{c}{b_2}}, c\right).$$

- (iii) *Let  $M \in \mathbb{Z}_{\geq 0}^n$  and  $m := |M|$ . Recall that  $\mathcal{D}_M \in \text{Diff}_K^{\leq m}(\hat{R})$  denotes the differential operator defined by  $\mathcal{D}_M(C_D t^D) = \binom{D}{M} C_D t^{D-M}$ . We set  $\mathcal{D}_M^{\log} := t^M \mathcal{D}_M \in \text{Diff}_K^{\leq m}(\hat{R})$ . Then*

$$\Delta((J, b) \cap (\mathcal{D}_M^{\log} J, b - m)) = \Delta(J, b).$$

*Proof.* The proofs emerge from a study of the vertices' behavior under the equivalences  $(J, b) \sim (J^a, ab)$ ,  $(J_1, b_1) \cap (J_2, b_2) \sim (J_1^{\frac{c}{b_1}} + J_2^{\frac{c}{b_2}}, c)$ , and  $(J, b) \cap (\mathcal{D}_M^{\log} J, b - m) \sim (J, b)$ . For a detailed proof see [Sc1], Lemma 2.4.4 and Lemma 2.4.5.  $\square$

Let  $\mathbb{E}$  be a pair on  $Z$  and  $x \in \text{Sing}(\mathbb{E})$ . Further denote by  $(u, y)$  a r.s.p. of  $\mathcal{O}_{Z, x}$  such that  $(y)$  determines  $\text{Dir}_x(\mathbb{E})$ . Then we define

$$\delta_x(\mathbb{E}, u) := \delta_x(\Delta(\mathbb{E}; u)) = \min\{|v| = v_1 + \dots + v_e \mid v \in \Delta(\mathbb{E}; u)\} \in \frac{1}{b!} \mathbb{Z}_{>1}^e$$

By the results of the previous sections we have

**Theorem 6.3** (Theorem D). *The rational number  $\delta_x(\mathbb{E}, u)$  does not depend on  $(y)$  and is invariant under the equivalence relation  $\sim$ . Therefore  $\delta_x(\mathbb{E}, u)_\sim$  is an invariant of the idealistic exponent  $\mathbb{E}_\sim$  and  $(u)$ .*

*Proof.* By definition  $\delta_x(\mathbb{E}, u)$  does not depend on  $(y)$ . By Theorem 5.5 it is attained by some  $(\hat{y})$  living in the completion of the local ring at  $x$ . Proposition 4.7 implies then the invariance under  $\sim$ .  $\square$

If we drop the assumption on  $(y)$  to give the directrix, then we do not know if there is a polyhedron which is independent of the system  $(y)$ ; we have shown in Example 5.6 that we are not able to make  $\Delta(f, b; u, y)$  independent of this choice.

But still we can say something in the case, where  $(y) = (y_1, \dots, y_s)$  can only be extended to a system  $(y_1, \dots, y_r), r > s$ , which yields the directrix:

**Lemma 6.4.** *Let  $\mathbb{E} = (J, b)$  be a pair on  $Z$  and  $x \in \text{Sing}(\mathbb{E})$  as before (thus  $\text{ord}_x(J) \geq b$ ). Fix a system of elements  $(u_1, \dots, u_d)$  in  $R = \mathcal{O}_{Z,x}$  which can be extended to a r.s.p. for  $R$ . Let  $(y) = (y_1, \dots, y_s)$  be such an extension of  $(u)$ . Assume further that  $(y, u_{e+1}, \dots, u_d)$ ,  $e < d$ , gives the directrix  $\text{Dir}_x(\mathbb{E})$ . Then we have*

$$\delta_x(\Delta(\mathbb{E}; u_1, \dots, u_d; y_1, \dots, y_s)) = 1$$

In particular this is independent of the choice of  $(y)$  and invariant under  $\sim$ . Therefore it is an invariant only depending on the idealistic exponent  $\mathbb{E}_\sim$  and  $(u)$ .

*Proof.* By assumption there is an  $f \in J$  with  $\text{in}(f, b) \notin K[Y_1, \dots, Y_s]$ . Hence its expansion  $f = \sum_{(A,B)} C_{A,B} u^A y^B$  there is an  $(A, B)$  such that

$$C_{A,B} \neq 0, \quad |A| \neq 0 \quad \text{and} \quad |A| + |B| = b.$$

Since  $\text{IDir}_x(\mathbb{E}) = \langle Y_1, \dots, Y_s, U_{e+1}, \dots, U_d \rangle$ , we can choose  $(A, B)$  such that the corresponding monomial cannot be deleted by any coordinate changes. Then  $(A, B)$  yields in  $\Delta_x(\mathbb{E}; u_1, \dots, u_d; y_1, \dots, y_s)$  the point  $v := \frac{A}{b-|B|}$  with  $|v| = 1$ . Further  $\text{ord}_x(J) \geq b$  implies

$$\delta_x(\Delta(\mathbb{E}; u_1, \dots, u_d; y_1, \dots, y_s)) \geq 1.$$

Together this yields the assertion.  $\square$

A first application of the characteristic polyhedra of idealistic exponents and these results is given in [Sc2], where the author deduces the invariant of Bierstone and Milman for constructive resolution of singularities in characteristic zero purely by considering certain polyhedra and their projections.

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